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ST 2012

Scientific Computing II

Exercise 3

May 21, 2012

Tutorial: Multigrid for Anisotropic Equations

We have to solve the following discretised anisotropic two-dimensional Poisson equation on the unit interval with homgeneous Dirichlet boundary conditions:

$$\begin{aligned} u_{xx} + 10^{-4} u_{yy} &= 0 \text{ in }]0;1[^2, \\ u &= 0 \text{ at } \partial]0;1[^2. \end{aligned}$$

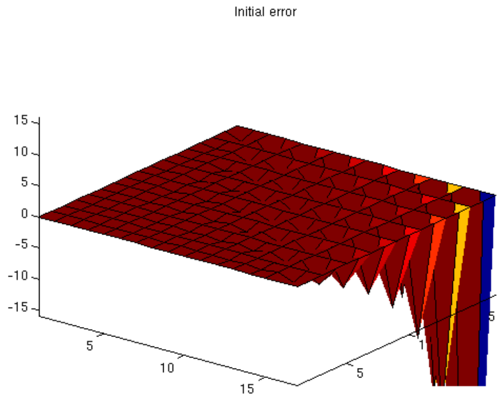
The anisotropic Laplacian is discretised on a regular cartesian grid by the 5-point-stencil

$$(u_{xx} + 10^{-4} u_{yy})(i \cdot h, j \cdot h) \approx \frac{10^{-4} u_{i,j-1} + u_{i-1,j} - (2 + 2 \cdot 10^{-4}) u_{i,j} + u_{i+1,j} + 10^{-4} u_{i,j+1}}{h^2}.$$

We want so solve the resulting discrete equation starting from the initial guess

$$u_{i,j} = \sum_{m,k=1}^{N} \sin(m\pi i h) \sin(k \pi j h)$$

with $N = \frac{1}{h} - 1$. That is, the initial guess is a combination of all possible error frequencies.



a) We use a multigrid method with

- standard coarsening,
- v-cycle,
- full-weighting as restriction operator,
- bilinear interpolation,
- two Gauss-Seidel pre- and postsmoothing iterations, each,

to solve the problem described above.

Describe the single steps of this multigrid algorithm.

b) Apply the method described above for $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{63}$ and report the resulting iteration numbers for a tolerance of 10^{-5} in the maximum norm of the residual in the following tabular:

n	3	7	15	31	63
# it					

c) You achieve a convergence behaviour similar to what method? What conclusions can you draw?

d) We use a multigrid method with

- semi-coarsening in x-direction,
- v-cycle,
- full-weighting as restriction operator,
- linear interpolation
- two Gauss-Seidel pre- and postsmoothing iterations, each,

to solve the problem described above.

Describe the single steps of this multigrid algorithm.

e) Apply the method described above for $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{63}$ and report the resulting iteration numbers for a tolerance of 10^{-5} in the maximum norm of the residual in the following tabular:

n	3	7	15	31	63
# it					

f) Perform task **e)** using line-smoothing instead of semi-coarsening in x-direction.

Hausaufgaben-Wichteln ("Secret Santa"): Multigrid

due by May 18, 12 am

- Hand in your solution (both plot and matlab code) per email to `eckhardw@in.tum.de` by Friday, 12 am, clearly stating which exercise you solved.
- In case you were not assigned a task in the tutorial, you can participate nevertheless: just choose one the tasks **a-e**!

We have to solve the discretised three-dimensional Poisson equation on a unit square with homgeneous Dirichlet boundary conditions (see exercise 2). The Laplacian is discretised by the known 7-point-stencil (see exercise 2).

a) We use the matlab multigrid solver from exercise 2 and compare the number of iterations required for different numbers n of pre- and m of postmoothing steps. The number of gridpoints N per coordinate direction is 127. What would be a reasonable size of $m+n$?

n	m	iterations
1	1	
2	2	
3	3	
4	4	
5	5	
1	0	
0	1	
3	0	
0	3	
5	0	
0	5	
7	0	
0	7	

b) We modify the multigrid v-cycle with 2 pre- and postsmoothing steps each by changing the coarsening strategy from a doubling of the stepsize h to a multiplication of h by a factor of 4:

$$H = 4 \cdot h$$

Compare the resulting iteration numbers to those achieved in exercise 1. What conclusion can you draw for the smoothing properties of Gauss-Seidel in this case?

Hint 1: Use the given functions `restrict_4h` and `interpolate_4h` for retriction and interpolation.

Hint 2: For a multiplication of h by a factor of 4 in each coarsening steps, the definition of high frequencies is different ($\theta_1, \theta_2 \in]-\pi; \pi] \setminus]-\frac{\pi}{4}; \frac{\pi}{4}]$) and, thus, the smoothing properties are different, too.

N	number of iterations
7	
15	
31	
63	
127	
255	

c) For those cases of **a)** for which you did not achieve a converging solver, try to replace injection by full weighting as a restriction operator. What results do you get? What does this mean for the suitability of injection as a restriction operator?

Hint: Use the given function `restrict_fw` for full weighting retriction.

m	n	iterations
1	0	
0	1	

d) For those cases of **a)** for which you did not achieve a converging solver, try to replace the V-cycle by a W-cycle. What results do you get?

m	n	iterations
1	0	
0	1	

e) Use the standard multigrid solver from exercise 2 (VCycle, 3 pre- and postsmoothing steps), but replace the Gauss-Seidel by an SOR solver with the optimal overrelaxation factor $\omega = \frac{2}{1+\sin(\pi h)}$. Record the number of iterations and runtime for the following different grids:

Multigrid method (V-cycle)

N	runtime	# iterations
7	sec	
15	sec	
31	sec	
63	sec	
127	sec	

Compare the number of iterations to the results obtained in task **a)**. What can you conclude?