

# Scientific Computing II

## Towards Multigrid Methods

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# Part I

## Multigrid Methods

### Fundamental Multigrid Ideas

- Nested Iteration

- Coarse-Grid Correction – A Two-Grid Method

### Multigrid Schemes

- Multigrid V-Cycle

- Multigrid W-Cycle

- Full Multigrid V-Cycle

### Computational Effort

- Costs per Iteration

- Speed of Convergence

# Multigrid Idea No. 1

Observation and convergence analysis show:

- “high-frequency error” is relative to mesh size
- on a sufficiently coarse grid, even very low frequencies can be “high-frequency”  
(if the mesh size is big)

“**Multigrid**” idea:

- use multiple grids to solve the system of equations
- on each grid, a certain range of error frequencies will be reduced efficiently

# Nested Iteration

Solve the problem on a coarser grid:

- will be comparably (very) fast
- can give us a good initial guess:
- “poor man’s multigrid”: **nested iteration**

## Algorithm:

1. Start on a very coarse grid with mesh size  $h = h_0$ ;  
guess an initial solution  $x_h$
2. Iterate over  $A_h x_h = b_h$  using **relaxation** method  
 $\Rightarrow$  approximate solution  $x_h$
3. **interpolate** the solution  $x_h$  to a finer grid  $\Omega_{h/2}$
4. proceed with step 2 (now with mesh size  $h := h/2$ ) using interpolated  $x_{h/2}$  as initial solution

## Multigrid Idea No. 2

Observation for nested iteration:

- error in interpolated initial guess also includes low frequencies
- relaxation therefore still slow
- can we go “back” to a coarser grid later in the algorithm?

Idea No. 2: use the residual equation

⇒ **coarse-grid correction:**

- solve  $Ae = r$  on a coarser grid
- leads to an approximation of the error  $e$
- add this approximation to the fine-grid solution

# A Two-Grid Method

Algorithm:

1. **relaxation/smoothing** on the fine level system  
 $\Rightarrow$  solution  $x_h$
2. compute the **residual**  $r_h = b_h - A_h x_h$
3. **restriction** of  $r_h$  to the coarse grid  $\Omega_H$
4. compute a **solution** to  $A_H e_H = r_H$
5. **interpolate** the coarse grid solution  $e_H$  to the fine grid  $\Omega_h$
6. add the resulting **correction** to  $x_h$
7. again, **relaxation/smoothing** on the fine grid

# Correction Scheme – Components

**smoother:** reduce the high-frequency error components, and get a smooth error

**restriction:** transfer residual from fine grid to coarse grid, for example by

- injection
- (full) weighting

**coarse grid equation:** (acts as) discretisation of the PDE on the coarse grid

**interpolation:** transfer coarse grid solution/correction from coarse grid to fine grid

# The Multigrid V-Cycle

Crucial idea: recursive call of Two-/Multigrid solver on coarse grids

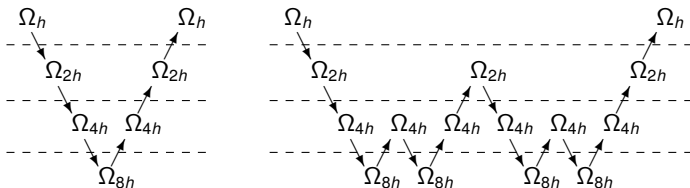
## Algorithm:

1. pre-smoothing on the fine level system  
 $\Rightarrow$  solution  $x_l$
2. compute the residual  $r_l = b_l - A_l x_l$
3. restriction of  $r_l$  to the coarse grid  $\Omega_{l-1}$
4. solve coarse grid system  $A_{l-1} e_{l-1} = r_{l-1}$   
by a **recursive call to the V-cycle algorithm**
5. interpolate the coarse grid solution  $e_{l-1}$  to the fine grid  $\Omega_l$
6. add the resulting correction to  $x_l$
7. post-smoothing on the fine grid



# The W-Cycle

- perform **two** coarse grid correction steps instead of one



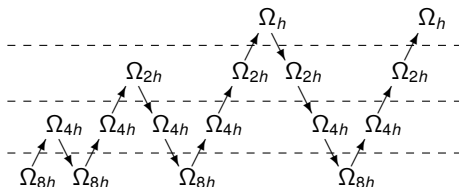
(V-cycle and W-cycle)

- more expensive
- useful in situations where the coarse grid correction is not very accurate

# The Full Multigrid V-Cycle (FMV)

Recursive algorithm:

- combines nested iteration and V-cycle
- (recursively!) perform an **FMV-cycle** on the next coarser grid to get a good initial solution
- interpolate this initial guess to the current grid
- perform a **V-cycle** to improve the solution



# V-Cycle – Computational Costs

Further algorithmic details:

- on the coarsest grid: direct solution;  
but: number of unknowns small, ideally  $\mathcal{O}(1)$
- number of smoothing steps is typically very small (1 or 2)  
and must not depend on problem size

**Computational Costs** (storage and computing time):

- 1D:  $c \cdot n + c \cdot n/2 + c \cdot n/4 + \dots \leq 2c \cdot n$
- 2D:  $c \cdot n + c \cdot n/4 + c \cdot n/16 + \dots \leq 4/3c \cdot n$
- 3D:  $c \cdot n + c \cdot n/8 + c \cdot n/64 + \dots \leq 8/7c \cdot n$
- overall costs are dominated by the costs of the finest grid

Thus: runtime  $\mathcal{O}(n)$  per iteration, but how many iterations necessary?

# Speed of Convergence

- fastest method around  
(if all components are chosen carefully)
- **“textbook multigrid efficiency”**:

$$\|e^{(m+1)}\| \leq \gamma \|e^{(m)}\|,$$

where convergence rate  $\gamma < 1$  (esp.  $\gamma \ll 1$ ) is independent of the number of unknowns

- ⇒ constant number of multigrid steps to obtain a given number of digits
- ⇒ overall computational work increases only linearly with the number of unknowns
- see exercises: analysis of two-grid convergence

## Speed of Convergence (2)

For the “Model Problem” (i.e., Poisson Problem):

- $\mathcal{O}(n)$  to solve up to “reduce error by a factor of ...” ( $10^{-8}$ , e.g.)
- wanted: solve to the “**level of truncation**”  
→ depends on discretisation error, for example  $\mathcal{O}(h^2)$
- $\mathcal{O}(n)$  up to “level of truncation” achieved by FMV-Cycle (1 cycle in the ideal case)

For Other Problems:

- OK for strongly elliptic problems
- multigrid variants for non-linear problems, parabolic/hyperbolic, ...
- every component may fail, leading to slow or no convergence: smoother, interpolation/restriction, coarse-grid operator
- achieving “textbook efficiency” usually a demanding task

## Part II

# Components of Multigrid Methods

**Smothers**  
**Interpolation**  
**Restriction**  
**Coarse Grid Operator**

# Smoothers

For the Poisson problem (see tutorials):

- Gauss-Seidel
- red-black Gauss-Seidel
- damped ( $\omega = \frac{2}{3}$ ) Jacobi
- how about Jacobi (non-weighted) and SOR?  
→ do not work well  
(do not smooth high frequencies efficiently)

# Smoothers – Anisotropic Problems

## Model problems and observation:

- anisotropic Poisson equation:  $u_{xx} + \epsilon u_{yy} = f$  with  $\epsilon \ll 1$   
(or similar:  $\epsilon \gg 1$ )
- Poisson equation on stretched grids:

$$\frac{1}{h_x^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{h_y^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{ij}$$

with  $h_x \ll h_y$  (or similar:  $h_x \gg h_y$ )

- Strong dependency in  $x$ -direction, weak dependency in  $y$ -direction (or vice versa)
- Good smoothing of the error only in  $x$ -direction  
(or in  $y$ -direction):

$$u_{i,j} = \frac{1}{2h_x^2 + 2h_y^2} \left( h_x^2 h_y^2 f_{ij} - h_y^2 (u_{i+1,j} + u_{i-1,j}) - h_x^2 (u_{i,j+1} + u_{i,j-1}) \right)$$



## Smoothers – Anisotropic Problems (2)

### Semi-Coarsening:

- situation on coarser grid for example  $H_x = 2h_x$ ,  $H_y = h_y$ :

$$\frac{1}{4h_x^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{1}{h_y^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) = f_{ij}$$

- thus: anisotropy has weakened on the semi-coarsened grid

### Line Smoothers:

- perform a column-wise Jacobi/Gauss-Seidel relaxation  
→ solve each “column” (or row) simultaneously:

$$u_{i-1,j}^{(n+1)} - (2 + 2\epsilon)u_{ij}^{(n+1)} + u_{i+1,j}^{(n+1)} = h^2 f_{ij} - \epsilon(u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)})$$

- use direct, tridiagonal solver for each “line”

# Smoothers – Convection-Diffusion

- example: 1D Convection-Diffusion equation

$$-\epsilon u_{xx} + u_x = f, \quad \epsilon \ll 1$$

- “upwind discretisation”:

$$-\frac{\epsilon}{h^2}(u_{n-1} - 2u_n + u_{n+1}) + \frac{1}{h}(u_n - u_{n-1}) = f_n$$

- (weighted) Jacobi and red-black Gauss-Seidel?  
→ no smoothing, basically updates one grid point per iteration
- Gauss-Seidel (relaxation from “left to right”)?  
→ almost an exact solver

# Smoothers – More Complicated Situations

## Problems:

- anisotropic Poisson with space-dependent  $\epsilon = \epsilon(x, y)$ , or more general:

$$-\nabla(D(x, y)\nabla u(x, y)) = f(x, y)$$

- convection-diffusion with variable convection:

$$-\epsilon u_{xx} + v(x)u_x = f \quad -\epsilon \Delta u + v(x, y)\nabla u(x, y) = f(x, y)$$

## Approaches for Smoothing:

- alternating line smoothers, “plane smoothers” in 3D
- “Zebra” line smoothers (similar to red-black-GS)
- Gauss-Seidel smoothing in “downwind” order  
→ difficult to do for complicated flows in 2D and 3D

# Interpolation (aka “Prolongation”)

## For Poisson problem:

- (bi-)linear interpolation:  
in 1D: resembles homogeneous ( $f = 0$ ) solution
- constant (in general too small approximation order):  
sometimes used for cell-based coarsening (unknowns located in cell centers)
- quadratic, cubic, etc.:  
often too costly, more smoothing steps are cheaper and can eliminate the disadvantage of a lower order interpolation
- **but:** in FMV-cycle interpolation to finer grid (after a completed V-cycle) should be higher-order

# Interpolation – Matrix Notation

For linear interpolation (1D):

$$\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(0 + x_1) \\ x_1 \\ \frac{1}{2}(x_1 + x_2) \\ x_2 \\ \frac{1}{2}(x_2 + x_3) \\ x_3 \\ \frac{1}{2}(x_3 + 0) \end{pmatrix}$$

Notation:  $I_{2h}^h x_{2h} = x_h$  or  $P_{2h}^h x_{2h} = x_h$

# Interpolation – Convection-Diffusion

**Example problem:** 1D convection-diffusion equation

$$-\epsilon u_{xx} + cu_x = f, \quad 0 < \epsilon \ll c$$

**Operator-dependent Interpolation:**

- consider homogeneous problem ( $f = 0$ )  
with Dirichlet boundaries:  $u(0) = 1$ ,  $u(1) = 0$
- exact solution for this case:

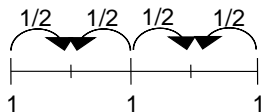
$$u(x) = \frac{1}{1 - e^{c/\epsilon}} \left( e^{cx/\epsilon} - e^{c/\epsilon} \right) = 1 - \frac{1 - e^{cx/\epsilon}}{1 - e^{c/\epsilon}}$$

- interpolate at  $x = \frac{1}{2}$ :  $u\left(\frac{1}{2}\right) = 1 - \frac{1 - e^{c/2\epsilon}}{1 - e^{c/\epsilon}} =: 1 - \frac{1-z}{1-z^2}$ ;  
for large  $z = \frac{c}{2\epsilon}$ , we have  $u\left(\frac{1}{2}\right) \approx 1 - \frac{1}{z}$
- thus: linear interpolation inappropriate (and leads to slow convergence)  $\rightarrow$  interpolation should be **operator-dependent**

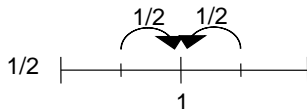
# Restriction

For Poisson problem:

- “injection”: pick values at corresp. coarse grid points
- “full weighting” = transpose of bilinear interpolation (safer, more robust convergence), see illustration below for the 1D case



linear interpolation



full weighting

# Restriction – Matrix Notation

For full weighting (1D):

$$\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_1 + 2x_2 + x_3) \\ \frac{1}{2}(x_3 + 2x_4 + x_5) \\ \frac{1}{2}(x_5 + 2x_6 + x_7) \end{pmatrix}$$

Notation:  $I_h^{2h} x_h = x_{2h}$  or  $R_h^{2h} x_h = x_{2h}$



# Coarse Grid Operator

Two main options:

1. discretise PDE on grid  $\Omega_h$  to obtain  $A_h$
2. “Galerkin approach”:  $A_{2h} := R_h^{2h} A_h P_{2h}^h$   
 → compare effect on vector  $x_{2h}$ :

$$A_{2h} x_{2h} := R_h^{2h} A_h P_{2h}^h x_{2h}$$

→ evaluate from right to left:

- interpolate  $x_{2h}$  to  $\hat{x}_h := P_{2h}^h x_{2h}$
- apply fine-grid operator  $A_h$  to interpolated  $\hat{x}_h$
- restrict resulting matrix-vector product to  $\Omega_{2h}$

## Exercise:

- Compute  $A_{2h} := R_h^{2h} A_h P_{2h}^h$  for  $A_h := \frac{1}{h^2} \text{tridiag}(-1, 2, -1)$

# A Matrix-Oriented View of Coarse-Grid Correction

1. given a system of equations  $A_h x_h = b_h$
2. pre-smoothing leads to approximate solution  $x_h^{(i)}$   
and resp. error  $x_h = x_h^{(i)} + e_h^{(i)}$
3. compute residual  $r_h^{(i)} = b_h - A_h x_h^{(i)}$ ;  
respective residual equation:  $A_h e_h^{(i)} = r_h^{(i)}$
4. restriction of residual equation:  $R_h^{2h} A_h e_h^{(i)} = R_h^{2h} r_h^{(i)}$
5. approximate error on coarse grid:  $e_h^{(i)} \approx P_{2h}^h e_{2h}^{(i)}$   
→ leads to Galerkin coarsening  $R_h^{2h} A_h P_{2h}^h e_{2h}^{(i)} = R_h^{2h} r_h^{(i)}$
6. compute error  $e_{2h}^{(i)}$  from  $(R_h^{2h} A_h P_{2h}^h) e_{2h}^{(i)} = A_{2h} e_{2h}^{(i)} = b_{2h} := R_h^{2h} r_h^{(i)}$
7. interpolate coarse-grid error,  $e_h^{(i)} \approx P_{2h}^h e_{2h}^{(i)}$ ,  
and apply as coarse-grid correction:  $x_h^{(i+1)} = x_h^{(i)} + P_{2h}^h e_{2h}^{(i)}$
8. post-smoothing on approximate solution  $x_h^{(i+1)}$

# Galerkin Coarse Grid Operator: $A_{2h} := R_h^{2h} A_h P_{2h}^h$

- assume linear Interpolation and full weighting restriction:

$$R_h^{2h} = \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \quad P_{2h}^h = (R_h^{2h})^T$$

- for Poisson equation with stencil  $\frac{1}{h^2}[-1 \ 2 \ -1]$  (see exercises)  
 → coarse-grid stencil reproduces:  $\frac{1}{(2h)^2}[-1 \ 2 \ -1]$
- for pure convection and central differencing, stencil  $\frac{1}{h}[1 \ 0 \ -1]$ ,  
 → coarse-grid stencil reproduces:  $\frac{1}{2h}[1 \ 0 \ -1]$  (check!)  
 → but leads to unstable discretisation
- question: result for upwind discretisation: stencil  $\frac{1}{h}[-1 \ 1 \ 1]$ ?

## Galerkin Coarsening and Convection

- coarsening with linear Interpolation, full weighting restriction, and upwind stencil  $\frac{1}{h} [-1 \quad 1 \quad 0]$
- leads to coarse-grid stencil  $\frac{1}{h} [-\frac{3}{4} \quad \frac{1}{2} \quad \frac{1}{4}]$ 
  - not a diagonal-dominant matrix
  - unstable discretisation
- remedy: use “downwind” interpolation and “upwind” restriction:

$$R_h^{2h} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad P_{2h}^h = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

- result: upwind discretisation stencil reproduced:  $\frac{1}{2h} [-1 \quad 1 \quad 0]!$

## Matrix-Dependent Interpolation/Restriction

- assume general 1D 3-point discretisation stencil:  $[s_l \quad s_c \quad s_r]$   
where  $s_c = -(s_l + s_r) > 0$
- use the following 3-point interpolation stencil:  $]-\frac{s_l}{s_c} \quad 1 \quad -\frac{s_r}{s_c}[$   
and  $P_{2h}^h = (R_h^{2h})^T$ , thus:

$$R_h^{2h} = \begin{pmatrix} -\frac{s_l}{s_c} & 1 & -\frac{s_r}{s_c} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{s_l}{s_c} & 1 & -\frac{s_r}{s_c} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{s_l}{s_c} & 1 & -\frac{s_r}{s_c} \end{pmatrix}$$

- Galerkin coarsening  $A_{2h} := R_h^{2h} A_h P_{2h}^h$  leads to stencil

$$\frac{1}{c} \begin{bmatrix} -l^2 & l^2 + r^2 & -r^2 \end{bmatrix}$$

( $\rightarrow$  check this computation!)

- apparently diagonal dominant  $\rightarrow$  stable coarse-grid discretisation
- try as exercise: compare operator- and matrix-dependent interpolation/restriction and coarsening for convection-diffusion equation

# Part III

## Finite Elements, Hierarchical Bases, and Multilevel Methods

### FEM Main Ingredients

Test and Shape Functions

1D Poisson Equation and Nodal Basis

### Hierarchical Bases and Generating Systems

Hierarchical Basis

FEM with Hierarchical vs. Nodal Bases

Hierarchical Basis Transformation

FEM and Hierarchical Basis Transform

### Generating Systems and Multigrid

Hierarchical Generating System and FEM

# Remember: Finite Elements – Main Ingredients

1. solve *weak form* of PDE to reduce regularity properties

$$u'' = f \quad \longrightarrow \quad \int v' u' \, dx = \int v f \, dx$$

→ allows additional *weak* solutions

2. compute a *function* as numerical solution

→ search in a function space  $W_h$ :

$$u_h = \sum_j u_j \varphi_j(x), \quad \text{span}\{\varphi_1, \dots, \varphi_J\} = W_h$$

3. find weak solutions of simple form:  
for example piecewise linear functions and choose basis functions with *local support* (“hat functions”)  
→ leads to system of linear equations

# Test and Shape Functions

- consider a general PDE  $Lu = f$  on some domain  $\Omega$
- search for solution functions  $u_h$  of the form

$$u_h = \sum_j u_j \varphi_j(x)$$

the  $\varphi_j(x)$  are typically called **shape** or **ansatz** functions

- the basis functions  $\varphi_j(x)$  build a vector space (i.e., a **function space**)  $W_h$

$$\text{span}\{\varphi_1, \dots, \varphi_J\} = W_h$$

- insert into weak formulation

$$\int vL\left(\sum_j u_j \varphi_j(x)\right) dx = \int vf dx \quad \forall v \in V$$



## Test and Shape Functions (2)

- choose a basis  $\{\psi_i\}$  of the *test space*  $V_h$   
typically defined on some discretisation grid  $\Omega_h$
- then: if all basis functions  $\psi_i$  satisfy

$$\int \psi_i(x) L\left(\sum_j u_j \varphi_j(x)\right) dx = \int \psi_i(x) f(x) dx \quad \forall \psi_i$$

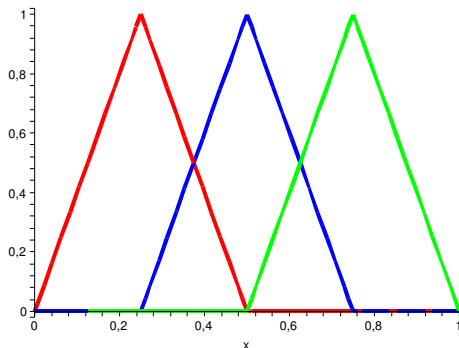
then all  $v \in V_h$  satisfy the equation

- the  $\{\psi_i\}$  are therefore often called **test functions**
- we obtain a system of equations for unknowns  $u_j$ :  
one equation for each test function  $\psi_i$
- $V_h$  is often chosen to be identical to  $W_h$  (**Ritz-Galerkin method**)  
→ we then have as many equations as unknowns
- leads to system of linear equations  $Au = b$  where

$$(Au)_i := \sum_j u_j \underbrace{\int \psi_i(x) L\varphi_j(x) dx}_{= A_{ij}} = \int \psi_i(x) f(x) dx =: b_i$$

# Example: Nodal Basis

$$\varphi_i(x) := \begin{cases} \frac{1}{h}(x - x_{i-1}) & x_{i-1} < x < x_i \\ \frac{1}{h}(x_{i+1} - x) & x_i < x < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$



## Example Problem: 1D Poisson

- in 1D:  $-u''(x) = f(x)$  on  $\Omega = (0, 1)$ ,  
hom. Dirichlet boundary cond.:  $u(0) = u(1) = 0$
- weak form:

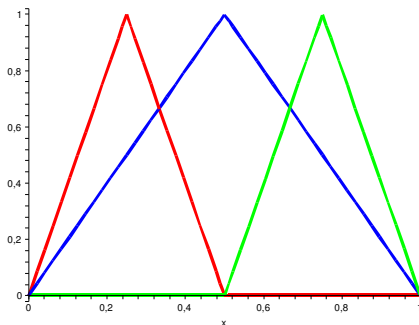
$$\int_0^1 v'(x) \cdot u'(x) \, dx = \int_0^1 v(x) f(x) \, dx \quad \forall v$$

- grid points  $x_i = ih$ , (for  $i = 1, \dots, n-1$ ); mesh size  $h = 1/n$
- $V_h = W_h$ : piecewise linear functions (on intervals  $[x_i, x_{i+1}]$ )
- leads to stiffness matrix:

$$\frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & -1 & 2 \\ & & & & -1 & 2 \end{pmatrix}$$

# Hierarchical Basis

- hat functions with multi-level resolution



- FEM solution identical for hierarchical and nodal basis (same function space!)
- however: **diagonal stiffness matrix** for 1D Poisson!

# Hierarchical vs. Nodal Basis

- consider mesh size  $h_n = 2^{-n}$  and grid points  $x_{n,i} = i \cdot h_n$
- Define “mother of all hat functions”

$$\phi(x) := \max\{1 - |x|, 0\}$$

- nodal basis then  $\Phi_n := \{\phi_{n,i}, 0 \leq i \leq 2^n\}$  with

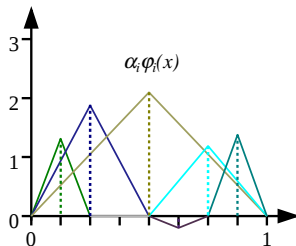
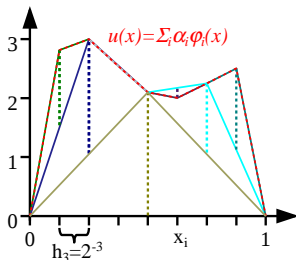
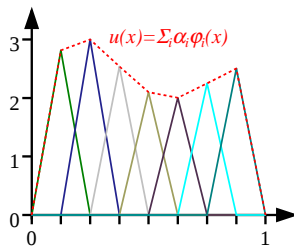
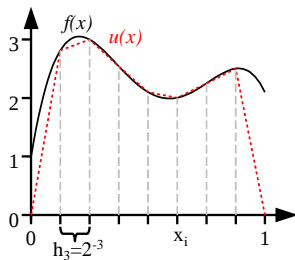
$$\phi_{n,i}(x) := \phi\left(\frac{x - x_{n,i}}{h_n}\right)$$

- hierarchical basis combines  $\widehat{\Phi}_n := \{\phi_{n,i}, i = 1, 3, \dots, 2^n - 1\}$  (only odd indices) and defines basis as

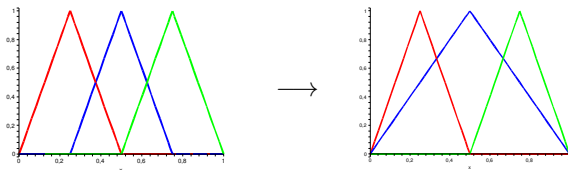
$$\Psi_n := \bigcup_{l=1}^n \widehat{\Phi}_l$$

- both span the same function space  $V_n := \text{span}(\Phi_n) = \text{span}(\Psi_n)$

# Hierarchical vs. Nodal Basis (2)



# Hierarchical Basis Transformation



- represent hat functions  $\phi_{n-1,i}(x)$  via fine-level functions  $\phi_{n,j}(x)$

$$\phi_{n-1,i}(x) = \frac{1}{2}\phi_{n,2i-1}(x) + \phi_{n,2i}(x) + \frac{1}{2}\phi_{n,2i+1}(x)$$

- hierarchical-basis transformation as matrix-vector product:

$$\begin{pmatrix} \psi_{n,i-1}(x) \\ \psi_{n,i}(x) \\ \psi_{n,i+1}(x) \end{pmatrix} := \begin{pmatrix} \phi_{n,2i-1}(x) \\ \phi_{n-1,i}(x) \\ \phi_{n,2i+1}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi_{n,2i-1}(x) \\ \phi_{n,2i}(x) \\ \phi_{n,2i+1}(x) \end{pmatrix}$$

## Hierarchical Basis Transformation (2)

- hierarchical basis transformation:  $\psi_{n,j}(x) = \sum_j H_{i,j} \phi_{n,j}(x)$
- written as matrix-vector product:  $\vec{\psi}_n = H_n \vec{\phi}_n$
- $H$  can be written as a sequence of level-wise transforms:

$$H_n = H_n^{(1)} H_n^{(2)} \dots H_n^{(n-2)} H_n^{(n-1)}$$

- where each transform has a shape similar to

$$H_3^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

cmp. restriction operator!



# Hierarchical Coordinate Transformation

- consider function  $f(x) \approx \sum_i a_i \psi_{n,i}(x)$  represented via hier. basis
- wanted: corresponding representation in nodal basis

$$\sum_k b_k \phi_{n,k}(x) = \sum_i a_i \psi_{n,i}(x) \approx f(x)$$

- with  $\psi_{n,i}(x) = \sum_j H_{i,j} \phi_{n,j}(x)$  we obtain

$$\sum_k b_k \phi_{n,k}(x) = \sum_i a_i \sum_j H_{i,j} \phi_{n,j}(x) = \sum_j \sum_i a_i H_{i,j} \phi_{n,j}(x)$$

- compare coordinates (identify indices  $j$  and  $k$ ) and get

$$b_j = \sum_i H_{i,j} a_i = \sum_i (H^T)_{j,i} a_i$$

- written in vector notation:  $b = H^T a$

# FEM and Hierarchical Basis Transform

- FEM discretisation with hierarchical test and shape functions:

$$\int \psi_i(\mathbf{x}) L \left( \sum_j v_j \psi_j(\mathbf{x}) \right) dx = \int \psi_i(\mathbf{x}) f(\mathbf{x}) dx \quad \forall \psi_i$$

- stiffness matrix with hierarchical basis as shape functions:

$$\int \psi_i(\mathbf{x}) L \left( \sum_j u_j \psi_j(\mathbf{x}) \right) dx = \sum_j u_j \int \psi_i(\mathbf{x}) L \psi_j(\mathbf{x}) dx = \sum_j u_j \mathbf{A}_{i,j}^{\text{HB}}$$

- vs. stiffness matrix with nodal basis as shape functions:

$$\int \psi_i(\mathbf{x}) L \left( \sum_j v_j \phi_j(\mathbf{x}) \right) dx = \sum_j v_j \int \psi_i(\mathbf{x}) L \phi_j(\mathbf{x}) dx = \sum_j v_j \mathbf{A}_{i,j}^*$$

- note that  $(\mathbf{A}^{\text{HB}} u)_i = \sum_j u_j \mathbf{A}_{i,j}^{\text{HB}} = \sum_j v_j \mathbf{A}_{i,j}^* = (\mathbf{A}^* v)_i$  and  $v = H^T u$

## FEM and Hierarchical Basis Transform (2)

- status: FEM with hierarchical test and nodal shape functions

$$\int \psi_i(x) L\left(\sum_j v_j \phi_j(x)\right) dx = \int \psi_i(x) f(x) dx$$

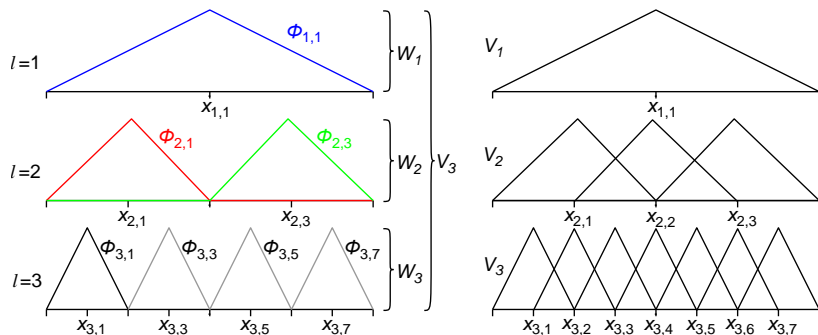
- represent test functions via nodal basis:

$$\int \sum_k H_{i,k} \phi_k(x) L\left(\sum_j v_j \phi_j(x)\right) dx = \int \sum_k H_{i,k} \phi_k(x) f(x) dx$$

$$\sum_k H_{i,k} \int \phi_k(x) L\left(\sum_j v_j \phi_j(x)\right) dx = \sum_k H_{i,k} \int \phi_k(x) f(x) dx$$

- leads to new system of equations:  $HA^{\text{NB}} v = Hb^{\text{NB}}$   
where  $A^{\text{NB}}$  and  $b^{\text{NB}}$  stem from nodal-basis FEM discretisation!
- with  $v = H^T u$  we obtain  $HA^{\text{NB}} H^T u = Hb$  as system of equations, thus:  $A^{\text{HB}} = HA^{\text{NB}} H^T$  ( $\rightsquigarrow$  **Galerkin coarsening**)

# Hierarchical Generating System



- hierarchical basis combines  $\widehat{\Phi}_n := \{\phi_{n,i}, i = 1, 3, \dots, 2^n - 1\}$  (only odd indices)
- generating system combines nodal bases of all levels:  $\bigcup_{l=1}^n \Phi_l$

# FEM and Hierarchical Generating Systems

- solution function  $u_h$  represented as

$$u_h = \sum_{l=1}^n \sum_{j=1}^{2^n-1} v_{l,j} \phi_{l,j}(x)$$

→ non-unique “multi-grid” representation!

- FEM discretisation with test and shape functions from hierarchical generating system:

$$\int \phi_{k,i}(x) L \left( \sum_l \sum_j v_{l,j} \phi_{l,j}(x) \right) dx = \int \phi_{k,i}(x) f(x) dx \quad \forall \phi_{k,i} \in \bigcup_{k=1}^n \Phi_k$$

- assume order of test/shape functions:

$$\phi_{n,1}, \dots, \phi_{n,2^n-1}, \phi_{n-1,1}, \dots, \phi_{n-1,2^{n-1}-1}, \dots, \phi_{2,1}, \phi_{2,2}, \phi_{2,3}, \phi_{1,1}$$

- leads to (linearly dependent!) system of linear equations:

$$A^{\text{GS}} u_h = b^{\text{GS}}$$

# Multi-Level System of Equations

- system matrix  $A^{\text{GS}}$  given as (for example)

$$\begin{pmatrix} A_h & A_h^{2h} & A_h^{4h} \\ A_{2h}^h & A_{2h} & A_{2h}^{4h} \\ A_{4h}^h & A_{4h}^{2h} & A_{4h} \end{pmatrix}$$

- $A_h$ ,  $A_{2h}$ , and  $A_{4h}$  are the nodal-basis stiffness matrices for resolution  $h$ ,  $2h$ , and  $4h$
- consider submatrix  $A_{2h}^h \rightarrow$  computed as

$$\int \phi_{2h,i}(x) L\left(\sum_j v_{h,j} \phi_{h,j}(x)\right) dx = \int \phi_{2h,i}(x) f(x) dx \quad \forall \phi_{2h,i} \in \Phi_{2h}$$

- test functions  $\phi_{2h,i}(x) = \frac{1}{2}\phi_{h,2i-1}(x) + \phi_{h,2i}(x) + \frac{1}{2}\phi_{h,2i+1}(x)$
- not a hierarchical transform, but a **restriction** operation  $R_h^{2h}$ :

$$\vec{\phi}_{2h} = R_h^{2h} \vec{\phi}_h \quad \text{thus:} \quad A_{2h}^h = R_h^{2h} A_h$$

## Multi-Level System of Equations (2)

- system of linear equations  $A^{\text{GS}} v^{\text{GS}} = b^{\text{GS}}$  thus given as

$$\begin{pmatrix} A_h & A_h P_{2h}^h & A_h P_{4h}^h \\ R_h^{2h} A_h & A_{2h} & A_{2h} P_{4h}^{2h} \\ R_h^{4h} A_h & R_{2h}^{4h} A_{2h} & A_{4h} \end{pmatrix} \begin{pmatrix} v_h \\ v_{2h} \\ v_{4h} \end{pmatrix} = \begin{pmatrix} b_h \\ R_h^{2h} b_h \\ R_h^{4h} b_h \end{pmatrix}$$

with restriction and prolongation operators:  $R = P^T$

- matrix is singular(!) due to obviously linearly dependent rows
- however: all solutions lead to same piecewise linear function
- result of a relaxation method on this system of equations?

**symmetric Gauß-Seidel sweep corresponds to one multigrid V-Cycle!**

(Griebel, 1994)

## Symmetric Gauß-Seidel on $A^{\text{GS}} v^{\text{GS}} = b^{\text{GS}}$

- pick out second block row of the system  $A^{\text{GS}} v^{\text{GS}} = b^{\text{GS}}$ :

$$R_h^{2h} A_h v_h + A_{2h} v_{2h} + A_{2h} P_{4h}^{2h} v_{4h} = R_h^{2h} b_h$$

- Gauß-Seidel  $\rightarrow$  thus solve for unknowns in  $v_{2h}$ :

$$A_{2h} v_{2h} + A_{2h} P_{4h}^{2h} v_{4h}^{(\text{old})} = R_h^{2h} b_h - R_h^{2h} A_h v_h^{(\text{new})} = R_h^{2h} (b_h - A_h v_h^{(\text{new})})$$

- observation #1: relaxation works on restricted fine-grid residual

$$R_h^{2h} (b_h - A_h v_h^{(\text{new})})$$

- observation #2: relaxation considers prolonged coarse-grid correction from previous iteration:  $A_{2h} P_{4h}^{2h} v_{4h}^{(\text{old})}$

$\Rightarrow$  **FEM on hierarchical generating systems matches V-Cycle Multigrid with Galerkin coarsening**



## Literature/References – Multigrid

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