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Sheet 1
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Scientific Computing II

Iterative Solvers

Exercise 1: Repetition "Finite Differences"

Consider the one-dimensional Poisson equation with homogeneous Dirichlet conditions

$$\begin{aligned} \frac{d^2u}{dx^2} &= f(x), \quad x \in (0,1), \\ u(0) = u(1) &= 0. \end{aligned} \tag{1}$$

(a) Discretise the Poisson equation by finite differences using an equidistant mesh size $h = 1/N$ and $N+1$ grid points.

(b) Write the finite difference approximation from (a) in matrix-vector form $Au = b$. Therefore, define the entries of the matrix $A \in \mathbb{R}^{N+1 \times N+1}$.

Solution:

(a) The discrete approximation reads

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i, \quad i = 1, ..., N-1 \tag{2}$$

where $u_i := u(ih)$, $f_i := f(ih)$.

(b) Eq. (2) corresponds to one line in the equation system; each line for inner points $i = 1,...,N-1$ is thus of the form

$$\begin{pmatrix} \dots & \frac{1}{h^2} & \frac{-2}{h^2} & \frac{1}{h^2} & \dots \end{pmatrix} \cdot \begin{pmatrix} \vdots \\ u_{i-1} \\ u_i \\ u_{i+1} \\ \vdots \end{pmatrix} = f_i. \tag{3}$$

The right hand side vector b thus corresponds to the function f, $b_i := f_i$. the boundary points yield an identity entry in the matrix and a right hand side $b_0 := u(0) = 0$, $b_N :=$

$u(1) = 0$. The matrix A thus arises as

$$A_{ij} := \begin{cases} 1 & \text{if} \quad (i,j) = (0,0),\ (i,j) = (N,N) \\ \frac{1}{h^2} & \text{if} \quad |i-j| = 1 \text{ and } i \neq 0, N \\ \frac{-2}{h^2} & \text{if} \quad i = j \text{ and } i \neq 0, N. \end{cases} \tag{4}$$

Row-Wise Derivation of Smoothers

Besides the matrix-based derivation (see lecture slides), most smoother methods can also be easily derived row-wise. Each row of the linear system reads:

$$\sum_j A_{ij} u_j = b_i, \quad i = 1, ..., N \tag{5}$$

Solving each equation for u_i results in

$$u_i = \frac{1}{A_{ii}} \left(b_i - \sum_{j \neq i} A_{ij} u_j \right). \tag{6}$$

- *Jacobi method*
  We solve the right hand side of Eq. (6) using the iterative solution at iteration step (n) and obtain the new iterative solution $(n+1)$:

$$u_i^{(n+1)} = \frac{1}{A_{ii}} \left(b_i - \sum_{j \neq i} A_{ij} u_j^{(n)} \right). \tag{7}$$

- *Weighted Jacobi method*
  We introduce a weighting factor ω and split the left hand side $u_i = \frac{1}{\omega} u_i + \left(1 - \frac{1}{\omega}\right) u_i$. We can now evaluate parts of u_i at (n) or $(n+1)$. We obtain:

$$\begin{aligned} \tfrac{1}{\omega} u_i^{(n+1)} + \left(1 - \tfrac{1}{\omega}\right) u_i^{(n)} &= \frac{1}{A_{ii}} \left(b_i - \textstyle\sum_{j \neq i} A_{ij} u_j^{(n)} \right) \\ \Leftrightarrow \qquad u_i^{(n+1)} &= \frac{\omega}{A_{ii}} \left(b_i - \textstyle\sum_{j \neq i} A_{ij} u_j^{(n)} \right) + (1 - \omega)\, u_i^{(n)}. \end{aligned} \tag{8}$$

  The right hand side of the last equation corresponds to a weighted average of the last solution $u_i^{(n)}$ and the solution predicted by the (non-weighted) Jacobi method.

- *Gauss-Seidel method*
  We solve the right hand side of Eq. (6) with both new and old values $u_j^{(n)}$, $u_j^{(n+1)}$. By this method, we can only use one array to store the solution u since we can immediately write the entries at $(n+1)$ into the original positions of the solutions $u_j^{(n)}$. The method reads:

$$u_i^{(n+1)} = \frac{1}{A_{ii}} \left(b_i - \sum_{j < i} A_{ij} u_j^{(n+1)} - \sum_{j > i} A_{ij} u_j^{(n)} \right). \tag{9}$$

Exercise 2: Fourier Analysis for Jacobi Methods

For a vanishing function $f(x) = 0$, the finite difference representation of Eq. (1) reduces to

$$u_{i-1} - 2u_i + u_{i+1} = 0 \tag{10}$$

The trivial solution of our problem with homogeneous boundary conditions is given by the solution $u = 0$. We can now investigate the behaviour of the smoothers such as Jacobi methods for this "toy problem".

(a) Formulate the Jacobi method for this case. Write the iteration scheme in the form

$$u_i^{(n+1)} = \sum_j M_{ij} u_i^{(n)}; \tag{11}$$

give an explicit representation of the iteration matrix M in this equation. Hint: use the formula for the iteration matrix $M := I - D_A^{-1}A$ from the lecture slides to construct the row-wise (matrix-free) update rule.

(b) Similar to Neumann stability analysis, we consider initial conditions $u_i^{(0)} = g_i^k := \sin(\pi k(ih))$ where $k \in \{0, ..., N-1\}$. What happens to the solution after one iteration step? Which convergence rate does this result invoke for the overall iteration for any initial condition g^k?

(c) Carry out the same analysis for the weighted Jacobi method with iteration matrix $M := I - \omega D_A^{-1}A$. What can you say in this case about the convergence rate?

Solution:

(a) One can compute the entries of M to be

$$M_{ij} = \begin{cases} \frac{1}{2} & \text{if} \quad |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

for the inner points of the system. The outer points are not affected since the boundary conditions are always retained; the iteration matrix thus has a row filled by zeros for the boundary points. We will therefore consider only the iteration for the inner points in the following. The iteration can hence be written as

$$u_i^{(n+1)} = \frac{1}{2}u_{i-1}^{(n)} + \frac{1}{2}u_{i+1}^{(n)} = \frac{1}{2}\left(u_{i-1}^{(n)} + u_{i+1}^{(n)}\right). \tag{13}$$

(b) Inserting the initial condition yields $u_i^{(1)}$ after one iteration:

$$\begin{aligned}
u_i^{(1)} &= \frac{1}{2}\left(u_{i-1}^{(n)} + u_{i+1}^{(n)}\right) \\
&= \frac{1}{2}\left(\sin(\pi k(i-1)h) + \sin(\pi k(i+1)h)\right) \\
&= \frac{1}{2}\big(\quad \sin(\pi kih)\cos(-\pi kh) + \sin(-\pi kh)\cos(\pi kih) \\
&\qquad + \sin(\pi kih)\cos(\ \ \pi kh) + \sin(\ \ \pi kh)\cos(\pi kih)\big) \\
&= \frac{1}{2}\cdot 2\cos(\pi kh)\sin(\pi kih) \\
&= \cos(\pi kh)\sin(\pi kih)
\end{aligned} \tag{14}$$

Note that we used the theorem $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$ in the third line of the transformation to re-write the sin-functions. Due to the symmetry of the cosine with respect to the y-axis and anti-symmetry of the sine, two terms cancel out and two other terms are identical (third to fourth line of the transformation).

Define $\mu_k := \cos(\pi kh)$. Since the factor μ_k is indendent from the coordinates ih and the iteration scheme is based on linear operations only, we can easily formulate the update rule recursively. Computing the respective update rule for an arbitrary iteration step n thus yields:

$$u_i^{(n)} = \mu_k u_i^{(n-1)} = \mu_k^2 u_i^{(n-2)} = ... = \mu_k^n \sin(\pi kih). \tag{15}$$

For any $k \in \{1, ...N-1\}$, the initial sine wave decreases by $|\mu_k| = |\cos(\pi kh)| < 1$ (for $k = 0$, the initial solution is already given by the vanishing sine wave $\sin(\pi \cdot 0 \cdot ih) = 0$). For $k = 1$, we see that the factor $\mu_1 \approx 1$. The solution is hence reduced very slowly. Similarly, for $k = N-1$, $\mu_{N-1} \approx -1$. The solution is thus also reduced slowly and always oscillates around zero.

(c) Setting up the update rule for the weighted Jacobi is established analogously to part (b). The update rule reads:

$$u_i^{(n+1)} = \frac{\omega}{2}\left(u_{i-1}^{(n)} + u_{i+1}^{(n)}\right) + (1-\omega)u_i^{(n)} \tag{16}$$

Re-using the theorem for $\sin(\alpha+\beta)$ yields

$$u_i^{(1)} = ((1-\omega) + \omega\cos(\pi kh))\sin(\pi kih) \tag{17}$$

with weighting parameter $\omega \in (0,1]$ for an initial condition $u_i^{(0)} = \sin(\pi kih)$. The factor μ_k is hence given by $\mu_k = (1-\omega) + \omega\cos(\pi kh)$ in this case. For $k = 1$, we still obtain $\mu_1 \approx 1$. However, when $k = N-1$, we obtain $\mu_{N-1} \approx (1-\omega) - \omega = 1 - 2\omega$. For a choice $\omega < \frac{1}{2}$, we hence obtain strictly positive rates, $\mu_k > 0$, and avoid oscillations ("damping").