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Scientific Computing II

Iterative Solvers

Exercise 3: Galerkin Construction of Coarse Grid Operator

We consider the fine grid given in Fig. 1 and want to solve a discrete Poisson system with homogeneous Dirichlet conditions on the unit interval,

$$\begin{aligned} -u_{i-1}^h + 2u_i^h - u_{i+1}^h &= h^2 f_i, \quad i = 1,2,3 \\ u_0^h = u_4^h &= 0, \end{aligned} \tag{1}$$

where the mesh size $h := \frac{1}{4}$. The system shall be denoted by $A_h u^h = b^h$ throughout the following.

(a) Define a linear mapping $R_h^{2h} : \mathbb{R}^5 \to \mathbb{R}^3$ according to the full weighting scheme (see Lecture slides). Setup the respective matrix R_h^{2h}.

(b) Define a linear mapping $P_{2h}^h : \mathbb{R}^3 \to \mathbb{R}^5$ which prolongates a solution vector from the coarse to the fine grid using linear interpolation (see Lecture slides). Setup the respective matrix P_{2h}^h.

(c) Based on the matrices R_h^{2h}, P_{2h}^h and A_h, compute the coarse grid operator $A_{2h} := R_h^{2h} A_h P_{2h}^h$ and compare it to A_h. What do you observe?

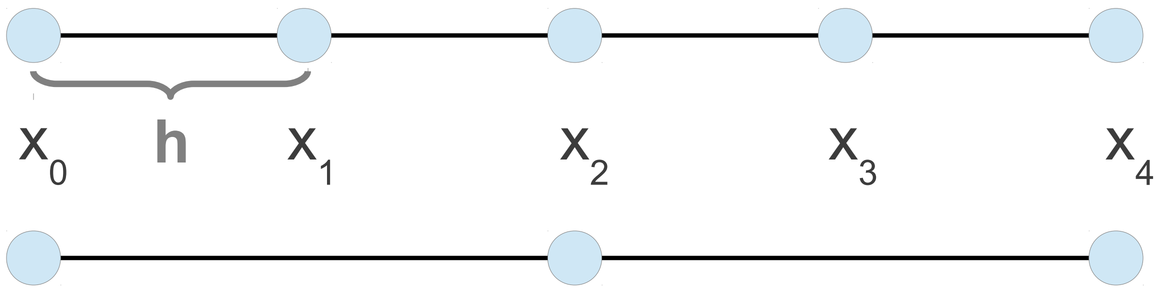

Figure 1: Fine (top) and coarse (bottom) grid.

Exercise 4: Fourier Analysis for Two-Grid Iteration

We stick to the Poisson example, see Eq. (1) from exercise 3. This time, however, we want to consider the more general case of having $N := 2K + 1$ grid points, and a resulting mesh size $h := 1/N$. A step towards multigrid methods consists in the *two-grid method* based on *coarse-grid correction*. In this particular case, we formulate the algorithm as follows:

- After some iteration steps using a smoother (like Jacobi method), compute residual $r^{old} := b^h - A_h u^h$.
  Remark: The residual and the error e^{old} fulfill the residual equation $r^{old} = Ae^{old}$.
- Restrict the residual to the coarse grid using an operator R. This yields a (residual-like) vector r^{2h} on the coarse grid. The coarse grid is assumed to have K points.
- Solve the residual equation $A_{2h}e^{2h} = r^{2h}$ on the coarse grid. A_{2h} corresponds to the coarse-grained system that resembles A_h on the fine grid.
- Interpolate the resulting coarse-grid approximation of the error e^{2h} to the fine grid and correct the fine-grid error $e^{new} := e^{old} - Pe^{2h}$. The interpolation operator shall be denoted by P.

Further define the different frequencies $q_h^m := (\sin(m\pi hi))_{i=1,\dots,N-1}$ on the fine grid. If $m < N/2$, we have a low frequency (with respect to the fine grid) and for $m \geq N/2$, q_h^m resembles a high frequency on the fine grid. The latter can thus not be represented on the coarse grid anymore.
In the following, we want to step through the Fourier analysis for this method and investigate the convergence behaviour, considering one cycle of this two-grid algorithm.

(a) Give a closed expression for e^{new} which only depends on e^{old} and not on e^{2h} anymore. You may use the operators R, P, A_{2h} and A_h from above.

(b) Define the restriction operator R by injection (see lecture slides). Show that the restriction of any error frequency Rq_h^m yields a low frequency, that is

$$Rq_h^m = \begin{cases} q_{2h}^m & \text{if} \quad m < N/2 \\ -q_{2h}^{N-m} & \text{if} \quad m \geq N/2. \end{cases} \tag{2}$$

(c) For interpolation, we want to use the linear interpolation scheme (analogous to exercise 3):

$$(Pq_{2h}^m)_i = \begin{cases} \underbrace{(q_{2h}^m)_{i/2}}_{= (q_h^m)_i} & \text{for} \quad i = 2, 4, \dots, N-2 \\ \frac{1}{2}\Big(\underbrace{(q_{2h}^m)_{(i-1)/2}}_{= (q_h^m)_{i-1}} + \underbrace{(q_{2h}^m)_{(i+1)/2}}_{= (q_h^m)_{i+1}}\Big) & \text{for} \quad i = 1, 3, \dots, N-1. \end{cases} \tag{3}$$

Use the function definitions

$$
\begin{aligned}
\tfrac{1}{2}(\cos(\pi i)+1) &= \begin{cases} 1 & \text{for } i = 2,4,\dots,N-2 \\ 0 & \text{for } i = 1,3,\dots,N-1 \end{cases} \\
\tfrac{1}{2}(-\cos(\pi i)+1) &= \begin{cases} 0 & \text{for } i = 2,4,\dots,N-2 \\ 1 & \text{for } i = 1,3,\dots,N-1 \end{cases}
\end{aligned} \tag{4}
$$

and the "trick"

$$
\underbrace{\sin((N-m)\pi hi)}_{=(q_h^{N-m})_i} = \underbrace{\sin(N\pi hi)}_{=0}\cos(m\pi hi) - \cos(N\pi hi)\underbrace{\sin(m\pi hi)}_{=(q_h^m)_i} = -\cos(N\pi hi) \tag{5}
$$

to re-write the interpolated coarse grid frequency Pq_{2h}^m as

$$
Pq_{2h}^m = a_m \cdot q_h^m + b_m \cdot q_h^{N-m} \tag{6}
$$

with (frequency-dependent) constants a_m, b_m. What does the latter equation tell us about the frequency of the interpolated function Pq_{2h}^m?

(d) Putting the results from prolongation and restriction together, we can derive by further computation (not discussed in the exercise) that

$$
Pe^{2h} = \begin{cases} q_h^m - \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} + q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \tag{7}
$$

Write the interpolated error Pe^{2h} as matrix-vector representation for the basis $\{q_h^m, q_h^{N-m}\}$ with $m < N/2$ and a 2×2-matrix B (we thus search for a 2×2-matrix which yields a mapping for coefficients x_1, x_2 when the initial frequency is given by $x_1 q_h^m + x_2 q_h^{N-m}$). Extend the matrix-vector form to the evaluation of the new error e^{new} on the fine grid,

$$
e^{new} = \underbrace{e^{old}}_{:=x_1 q_h^m + x_2 q_h^{N-m}} - Pe^{2h}. \tag{8}
$$

(e) In order to remove high frequency errors, we apply two Jacobi iterations before (*pre-smoothing*) and after (*post-smoothing*) the coarse-grid correction scheme. Following the Fourier analysis (see last exercise) for the basis vector $\{q_h^m, q_h^{N-m}\}$, one Jacobi iteration corresponds to the update rule

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{9}
$$

where x_1, x_2 are the coefficients for the subspace spanned by the basis $\{q_h^m, q_h^{N-m}\}$ from the previous sub-exercise. Give an estimate for the maximum eigenvalue of the overall solver operations, that is pre-smoothing $\rightarrow$ restriction $\rightarrow$ coarse-grid correction $\rightarrow$ interpolation $\rightarrow$post-smoothing.

Programming Exercise 1: Smoothing Properties of (Weighted) Jacobi Method

We want to numerically solve the discrete Poisson problem from Eq. (1) with arbitrary number of grid points $N+1$, a mesh size $h := 1/N$ and a right hand side $f_i = 0$ for all inner points. Together with homogeneous Dirichlet conditions, we thus expect our iteration procedure to converge towards $u_i = 0$ for all i.

(a) Use matlab to formulate one iteration of the Jacobi method using a pointwise update rule.

(b) Use initial values $u_i^k := \sin(\pi k i h)$, $k = 1, 3, 7$, at the grid points x_i and a mesh size $h := 1/8$. Extend the matlab script to carry out 10 relaxation steps. At which rate does the error $e_k^{(n)} := \max_i |u_i^{k(n)} - 0|$ decrease in each iteration step? Measure $r := e_k^{(n)}/e_k^{(n-1)}$ for this purpose. Compare your results with the analytical findings from exercise 2.

(c) What happens when decreasing the meshsize to $h := 1/16$ and keeping all other parameters unchanged?

(d) Carry out the same study for the weighted Jacobi method with $\omega = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$.