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Scientific Computing II

Iterative Solvers

Exercise 3: Galerkin Construction of Coarse Grid Operator

We consider the fine grid given in Fig. 1 and want to solve a discrete Poisson system with homogeneous Dirichlet conditions on the unit interval,

$$\begin{aligned} -u_{i-1}^h + 2u_i^h - u_{i+1}^h &= h^2 f_i, \quad i = 1,2,3 \\ u_0^h = u_4^h &= 0, \end{aligned} \tag{1}$$

where the mesh size $h := \frac{1}{4}$. The system shall be denoted by $A_h u^h = b^h$ throughout the following.

(a) Define a linear mapping $R_h^{2h} : \mathbb{R}^5 \to \mathbb{R}^3$ according to the full weighting scheme (see Lecture slides). Setup the respective matrix R_h^{2h}.

(b) Define a linear mapping $P_{2h}^h : \mathbb{R}^3 \to \mathbb{R}^5$ which prolongates a solution vector from the coarse to the fine grid using linear interpolation (see Lecture slides). Setup the respective matrix P_{2h}^h.

(c) Based on the matrices R_h^{2h}, P_{2h}^h and A_h, compute the coarse grid operator $A_{2h} := R_h^{2h} A_h P_{2h}^h$ and compare it to A_h. What do you observe?

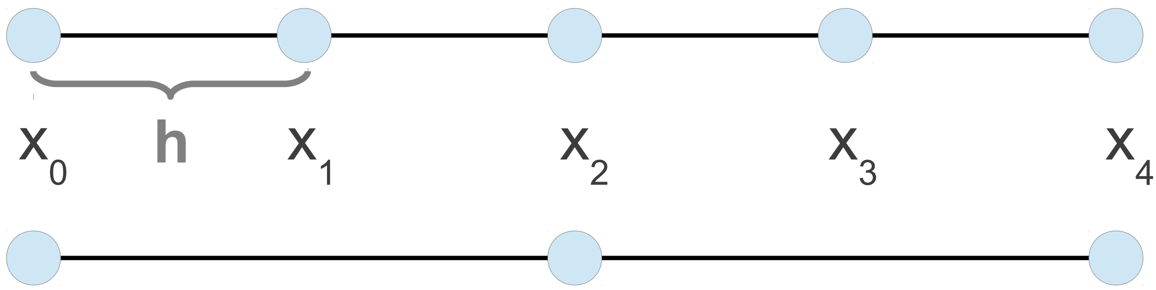

Figure 1: Fine (top) and coarse (bottom) grid.

Solution:

(a) The full weighting scheme uses weights (... 1/4 1/2 1/4 ...) to approximate the restricted vector $u^{2h} \in \mathbb{R}^3$. We obtain

$$R_h^{2h} := \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}. \tag{2}$$

One could also imagine to put a unit row for the Dirichlet boundary points (i.e. (1,0,0,0,0) for the left boundary). Although a respectively modified restriction matrix R_h^{2h} does not imply any changes in the overall operator $A_{2h} = R_h^{2h} A_h P_{2h}^h$, this restriction does not sound "fair" for all coarse grid points: in this example, the information of the residual vector at point x_1 would be only transported to x_2, but not to x_0 on the coarse grid which in turn yields an asymmetric behaviour.

(b) At the coarse grid points which coincide with respective fine grid points, the linear interpolation scheme only uses the values at these points. Otherwise, the fine grid values arise as the average of the values at the two neighbouring coarse grid points:

$$u^h(x_i) = \begin{cases} u^{2h}(x_i) & \text{if} \quad i = 0, 2, 4 \\ \frac{1}{2}(u^{2h}(x_{(i-1)/2}) + u^{2h}(x_{(i+1)/2}) & \text{if} \quad i = 1, 3. \end{cases} \tag{3}$$

The arising mapping P_{2h}^h reads:

$$P_{2h}^h := \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

(c) The explicit representation of matrix A_h is given by:

$$A_h := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

Evaluating the matrix product for A_{2h} yields:

$$A_{2h} = R_h^{2h} A_h P_{2h}^h = ... = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

For the only inner point x_2 on the coarse grid, we recognise that the form of the respective linear equation still remains the same as on the fine grid, except for a scaling factor 4 in this case. This factor arises from halving the mesh size, implying a change $2 \times 2 = 4$ in the overall stencil.

Remark: In the general Galerkin approach, one chooses $P_{2h}^{h} = {R_h^{2h}}^\top$. In the present example this corresponds to multiplying R_h^{2h} by a factor of 2. This yields the equality of prolongation and transposed restriction operator and does not impose any changes to the final result since we also need to apply the restriction to the right-hand side of our linear system of equations. The factor is thus a constant on both sides and can be chosen arbitrarily.

Exercise 4: Fourier Analysis for Two-Grid Iteration

We stick to the Poisson example, see Eq. (1) from exercise 3. This time, however, we want to consider the more general case of having $N := 2K + 1$ grid points, and a resulting mesh size $h := 1/N$. A step towards multigrid methods consists in the *two-grid method* based on *coarse-grid correction*. In this particular case, we formulate the algorithm as follows:

- After some iteration steps using a smoother (like Jacobi method), compute residual $r^{old} := b^h - A_h u^h$.
  Remark: The residual and the error e^{old} fulfill the residual equation $r^{old} = Ae^{old}$.
- Restrict the residual to the coarse grid using an operator R. This yields a (residual-like) vector r^{2h} on the coarse grid. The coarse grid is assumed to have K points.
- Solve the residual equation $A_{2h}e^{2h} = r^{2h}$ on the coarse grid. A_{2h} corresponds to the coarse-grained system that resembles A_h on the fine grid.
- Interpolate the resulting coarse-grid approximation of the error e^{2h} to the fine grid and correct the fine-grid error $e^{new} := e^{old} - Pe^{2h}$. The interpolation operator shall be denoted by P.

Further define the different frequencies $q_h^m := (\sin(m\pi hi))_{i=1,\dots,N-1}$ on the fine grid. If $m < N/2$, we have a low frequency (with respect to the fine grid) and for $m \geq N/2$, q_h^m resembles a high frequency on the fine grid. The latter can thus not be represented on the coarse grid anymore.
In the following, we want to step through the Fourier analysis for this method and investigate the convergence behaviour, considering one cycle of this two-grid algorithm.

(a) Give a closed expression for e^{new} which only depends on e^{old} and not on e^{2h} anymore. You may use the operators R, P, A_{2h} and A_h from above.

(b) Define the restriction operator R by injection (see lecture slides). Show that the restriction of any error frequency Rq_h^m yields a low frequency, that is

$$Rq_h^m = \begin{cases} q_{2h}^m & \text{if} \quad m < N/2 \\ -q_{2h}^{N-m} & \text{if} \quad m \geq N/2. \end{cases} \tag{7}$$

(c) For interpolation, we want to use the linear interpolation scheme (analogous to exercise 3):

$$(Pq_{2h}^m)_i = \begin{cases} \underbrace{(q_{2h}^m)_{i/2}}_{=(q_h^m)_i} & \text{for} \quad i = 2, 4, \dots, N-2 \\ \frac{1}{2}\big(\underbrace{(q_{2h}^m)_{(i-1)/2}}_{=(q_h^m)_{i-1}} + \underbrace{(q_{2h}^m)_{(i+1)/2}}_{=(q_h^m)_{i+1}}\big) & \text{for} \quad i = 1, 3, \dots, N-1. \end{cases} \tag{8}$$

Use the function definitions

$$\begin{aligned} \tfrac{1}{2}(\cos(\pi i)+1) &= \begin{cases} 1 & \text{for } i = 2,4,\ldots,N-2 \\ 0 & \text{for } i = 1,3,\ldots,N-1 \end{cases} \\ \tfrac{1}{2}(-\cos(\pi i)+1) &= \begin{cases} 0 & \text{for } i = 2,4,\ldots,N-2 \\ 1 & \text{for } i = 1,3,\ldots,N-1 \end{cases} \end{aligned} \tag{9}$$

and the "trick"

$$\underbrace{\sin((N-m)\pi hi)}_{=(q_h^{N-m})_i} = \underbrace{\sin(N\pi hi)}_{=0}\cos(m\pi hi) - \cos(N\pi hi)\underbrace{\sin(m\pi hi)}_{=(q_h^m)_i} = -\cos(N\pi hi) \tag{10}$$

to re-write the interpolated coarse grid frequency Pq_{2h}^m as

$$Pq_{2h}^m = a_m \cdot q_h^m + b_m \cdot q_h^{N-m} \tag{11}$$

with (frequency-dependent) constants a_m, b_m. What does the latter equation tell us about the frequency of the interpolated function Pq_{2h}^m?

(d) Putting the results from prolongation and restriction together, we can derive by further computation (not discussed in the exercise) that

$$Pe^{2h} = \begin{cases} q_h^m - \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} & \text{if} \quad m < \frac{N}{2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} q_h^{N-m} + q_h^m & \text{if} \quad m \geq \frac{N}{2} \end{cases} \tag{12}$$

Write the interpolated error Pe^{2h} as matrix-vector representation for the basis $\{q_h^m, q_h^{N-m}\}$ with $m < N/2$ and a 2×2-matrix B (we thus search for a 2×2-matrix which yields a mapping for coefficients x_1, x_2 when the initial frequency is given by $x_1 q_h^m + x_2 q_h^{N-m}$). Extend the matrix-vector form to the evaluation of the new error e^{new} on the fine grid,

$$e^{new} = \underbrace{e^{old}}_{:= x_1 q_h^m + x_2 q_h^{N-m}} - Pe^{2h}. \tag{13}$$

(e) In order to remove high frequency errors, we apply two Jacobi iterations before (*pre-smoothing*) and after (*post-smoothing*) the coarse-grid correction scheme. Following the Fourier analysis (see last exercise) for the basis vector $\{q_h^m, q_h^{N-m}\}$, one Jacobi iteration corresponds to the update rule

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{14}$$

where x_1, x_2 are the coefficients for the subspace spanned by the basis $\{q_h^m, q_h^{N-m}\}$ from the previous sub-exercise. Give an estimate for the maximum eigenvalue of the overall solver operations, that is pre-smoothing $\to$ restriction $\to$ coarse-grid correction $\to$ interpolation $\to$post-smoothing.

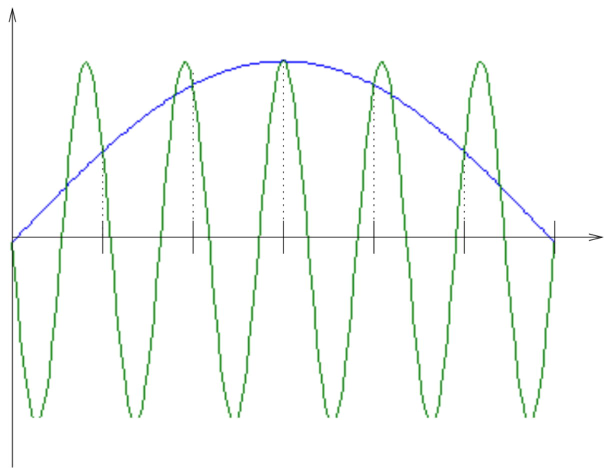

Figure 2: High-to-low frequency convertion when using restriction by injection.

Solution:

(a) The closed expression reads

$$\begin{aligned} e^{new} &= e^{old} - Pe^{2h} = e^{old} - PA_{2h}^{-1}r^{2h} = e^{old} - PA_{2h}^{-1}Rr^{old} = e^{old} - PA_{2h}^{-1}RA_h e^{old} \\ &= (I - PA_{2h}^{-1}RA_h)e^{old}. \end{aligned} \tag{15}$$

(b) If we apply restriction by injection, we can only resolve the frequency at every second grid point. This means that we can represent a frequency q_h^m on the coarse grid as

$$Rq_h^m = (\sin(m\pi hi))_{i=2,4,\dots,N-2} = (\sin(m\pi 2hi))_{i=1,\dots,\frac{N}{2}-1}. \tag{16}$$

For a low frequency $m < N/2$ which can still be resolved on the coarse grid, we can interprete the latter as frequency q_{2h}^m on the coarse grid nodes.
For higher frequencies $m \geq N/2$, we know that these cannot be resolved on the coarse grid anymore. Still, they should yield a contribution on the coarse grid. To understand their behaviour, we transform Eq. (**??**) using the sum formula for the sine:

$$\begin{aligned} \sin(m\pi 2hi) &= \sin((m - N + N)\pi 2hi) \\ &= \sin((m-N)\pi 2hi)\underbrace{\cos(N\pi 2hi)}_{=\cos(2\pi i)=1} + \cos((m-N)\pi 2hi)\underbrace{\sin(N\pi 2hi)}_{=\sin(2\pi i)=0} \\ &= -\sin((N-m)\pi 2hi). \end{aligned} \tag{17}$$

The latter term $\sin((N-m)\pi 2hi)$ corresponds to a low frequency again (since for $m \geq N/2$, we have $N - m \leq N/2$). This behaviour is also sketched in Fig. **??**. The restriction of a frequency q_h^m thus yields:

$$Rq_h^m = \begin{cases} q_{2h}^m & \text{if} \quad m < \frac{N}{2} \\ -q_{2h}^{N-m} & \text{if} \quad m \geq \frac{N}{2}. \end{cases} \tag{18}$$

(c) The cosine-based function definitions can be used to re-write the case distinction in a single formula using a linear combination of both cases:

$$\begin{aligned}(Pq_{2h}^m)_i &= \tfrac{1}{2}(\cos(\pi i)+1)(q_h^m)_i + \tfrac{1}{2}(-\cos(\pi i)+1)\tfrac{1}{2}\left((q_h^m)_{i-1}+(q_h^m)_{i+1}\right) \\ &= \tfrac{1}{2}(\cos(\pi i)+1)(q_h^m)_i + \tfrac{1}{2}(-\cos(\pi i)+1)\cos(m\pi h)(q_h^m)_i \\ &= \tfrac{1}{2}\left[(1+\cos(m\pi h))(q_h^m)_i + (1-\cos(m\pi h))\underbrace{\cos(\pi i)}_{=\cos(N\pi hi)}\underbrace{(q_h^m)_i}_{=\sin(m\pi hi)}\right]\end{aligned} \tag{19}$$

The step from the first to the second row of Eq. (**??**) involves again the sum formula for the sine (see also exercise 2 from the previous sheet). The last row of Eq. (**??**) only resembles a re-ordering of the terms.
In the last row of Eq. (**??**), we still have a coefficient $\cos(\pi i)$ which yields a spatial dependency of the frequency after interpolation. We remove this dependency by a similar trick as in the previous discussion of the restriction operator: we therefore consider again the frequency q_h^{N-m} which can be re-written as

$$\underbrace{\sin((N-m)\pi hi)}_{=(q_h^{N-m})_i} = \underbrace{\sin(N\pi hi)}_{=0}\cos(m\pi hi) - \cos(N\pi hi)\underbrace{\sin(m\pi hi)}_{=(q_h^m)_i}. \tag{20}$$

Inserting this relation into Eq. (**??**) yields

$$(Pq_{2h}^m)_i = \frac{1}{2}\left[(1+\cos(m\pi h))(q_h^m)_i + (-1+\cos(m\pi h))\underbrace{\sin((N-m)\pi hi)}_{(q_h^{N-m})_i}\right] \tag{21}$$

or in vector notation

$$Pq_{2h}^m = \frac{1}{2}\left[(1+\cos(m\pi h))q_h^m + (-1+\cos(m\pi h))q_h^{N-m}\right]. \tag{22}$$

From the latter equation, we can observe one interesting thing: if $m < N/2$, i.e. m corresponds to a low frequency, we will obtain a high frequency contribution $N-m$ on the fine grid. If $m \geq N/2$, we will obtain a high frequency contribution m on the fine grid. Consequently, no matter whether the frequency m corresponds to a rather high or low frequency mode, we will always generate a high frequency mode on the fine grid! Hence, after prolongation/ interpolation, we always need to apply *post-smoothing* to remove these generated high frequencies.

(d) For the initial error frequency q_h^m, we can obtain the expression given in Eq. (7) by explicitly resolving the operator sequence $Pe^{2h} = PA_{2h}^{-1}RA_hq_h^m$ (based on the results from the previous sub-exercises). We skip this lengthy computation at this point. Since the outcome only depends on q_h^m and q_h^{N-m}, we can formulate Pe^{2h} in matrix-vector form as follows:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{23}$$

where x_1, x_2 are coefficients for the subspace spanned by $\{q_h^m, q_h^{N-m}\}$.
For

$$e^{new} = e^{old} - Pe^{2h} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 1 & -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ -\frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{24}$$

this results in

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{25}$$

(e) Combining the result of the previous exercise with the pre- and post-smoothing, we obtain the following relation for the coefficients x_1, x_2 over one two-grid cycle:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \underbrace{\begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix}^2}_{\text{post-smoothing}} \begin{pmatrix} 0 & \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} \\ \frac{(1-\cos(m\pi h))^2}{\sin(m\pi h)^2} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \frac{1+\cos(m\pi h)}{2} & 0 \\ 0 & \frac{1-\cos(m\pi h)}{2} \end{pmatrix}^2}_{\text{pre-smoothing}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{26}$$

which after some further transformations can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \\ \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{27}$$

The eigenvalues of this matrix are given by

$$\begin{aligned} \lambda_1 &= \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \\ \lambda_2 &= -\frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16}. \end{aligned} \tag{28}$$

We can thus give an estimate of the maximum eigenvalue:

$$\max_m |\lambda_1| = \max_m |\lambda_2| = \max_m \left| \frac{\sin(m\pi h)^2(1-\cos(m\pi h))^2}{16} \right| \leq \frac{1}{16}. \tag{29}$$

Let's review the overall procedure so far: we inserted an error frequency q_h^m into our method, determined the transfer of this frequency from the fine to the coarse grid (restriction) and from the coarse to the fine grid (prolongation/ interpolation). Subsequently, we formulated the overall scheme for this frequency in the subspace of the basis vectors $\{q_h^m, q_h^{N-m}\}$ since this turned out to be sufficient to describe the frequency transfer. After including the pre- and post-smoothing, we could now give an estimate of the eigenvalues of our solver which determine how strong the error is decreased.
From the last equation, we can now see that the error is at least reduced by a factor of 16 in one two-grid cycle. This is even independent from the mesh size (unlike the smoothers discussed so far (such as Jacobi) where we saw that we need the more iterations the more grid points we use). In order to achieve a certain accuracy, this method consequently also does not depend on the resolution h and thus delivers an optimal convergence behaviour.

Programming Exercise 1: Smoothing Properties of (Weighted) Jacobi Method

We want to numerically solve the discrete Poisson problem from Eq. (1) with arbitrary number of grid points $N+1$, a mesh size $h := 1/N$ and a right hand side $f_i = 0$ for all inner points. Together with homogeneous Dirichlet conditions, we thus expect our iteration procedure to converge towards $u_i = 0$ for all i.

(a) Use matlab to formulate one iteration of the Jacobi method using a pointwise update rule.

(b) Use initial values $u_i^k := \sin(\pi k i h)$, $k = 1, 3, 7$, at the grid points x_i and a mesh size $h := 1/8$. Extend the matlab script to carry out 10 relaxation steps. At which rate does the error $e_k^{(n)} := \max_i |u_i^{k(n)} - 0|$ decrease in each iteration step? Measure $r := e_k^{(n)} / e_k^{(n-1)}$ for this purpose. Compare your results with the analytical findings from exercise 2.

(c) What happens when decreasing the meshsize to $h := 1/16$ and keeping all other parameters unchanged?

(d) Carry out the same study for the weighted Jacobi method with $\omega = \frac{1}{3}, \frac{2}{3}, \frac{1}{2}$.

Solution:

(a) The solution can be found in smoothers.m.

(b) In all simulations, we can observe a constant rate of error decrease. This rate exactly corresponds to the eigenvalue of our iterative matrix-vector operation, assuming a given input frequency. We obtain:

$$\begin{aligned} k = 1: \quad r = 0.9239 &= \cos(k\pi h) \\ k = 3: \quad r = 0.3827 &= \cos(k\pi h) \\ k = 7: \quad r = 0.9239 &= -\cos(k\pi h). \end{aligned} \tag{30}$$

The decay of the error is thus exactly as expected from our Fourier analysis in exercise 2 where we predicted a decrease of $|\cos(k\pi h)|$.

(c) Changing the grid resolution implies a frequency modification with respect to the grid resolution. We know from exercise 2, that frequencies at the very lower and upper end of the spectrum are removed very slowly. The frequency $k = 3$ is—with respect to the fine grid—now quite at the "lower" end of the frequency spectrum. We thus expect a worse convergence compared to the coarser grid. The frequency $k = 7$, however, is now in the middle of the spectrum. We therefore expect a very good damping of this error mode. We obtain the following rates in error reduction:

$$\begin{aligned} k = 1: \quad r = 0.9808 &= \cos(k\pi h) \\ k = 3: \quad r = 0.8315 &= \cos(k\pi h) \\ k = 7: \quad r = 0.1951 &= \cos(k\pi h). \end{aligned} \tag{31}$$

(d) For $\omega = 1/3, h = 1/8$, we obtain:

$$\begin{aligned} k = 1: \quad r = 0.9808 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 3: \quad r = 0.7942 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 7: \quad r = 0.3587 &= 1 - \omega + \omega\cos(k\pi h). \end{aligned} \tag{32}$$

For $\omega = 2/3, h = 1/8$, we obtain:

$$\begin{aligned} k = 1: \quad r = 0.9493 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 3: \quad r = 0.5885 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 7: \quad r = 0.2826 &= -(1 - \omega + \omega\cos(k\pi h)). \end{aligned} \tag{33}$$

If we consider the error plots of $(x_i, u_i^{k(n)})$, we observe that for $\omega = 1/3$, the error homogeneously decays without oscillating around the zero line. Since $\omega = 1/3 < 1/2$, this is what we already expected after the Fourier analysis in exercise 2 for the damped Jacobi method.

For $\omega = 1/2, h = 1/8$, we obtain:

$$\begin{aligned} k = 1: \quad r = 0.9619 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 2: \quad r = 0.8536 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 3: \quad r = 0.5913 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 4: \quad r = 0.5000 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 5: \quad r = 0.3087 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 6: \quad r = 0.1464 &= 1 - \omega + \omega\cos(k\pi h) \\ k = 7: \quad r = 0.0381 &= 1 - \omega + \omega\cos(k\pi h). \end{aligned} \tag{34}$$

Due to the monotony and the strict positivity for the weights $\omega = 1/3, 1/2$, we obtain an improvement of the error reduction rates for increasing frequencies k. We can thus obtain very good damping of all high frequencies by tuning the weighting parameter ω. Using $\omega = 1/2$ instead of $\omega = 1/3$ thus yields better reduction of the high frequencies in the Poisson problem.