Technische Universität München
Lehrstuhl für Informatik V
Univ.-Prof. Dr. M. Bader
M.Sc.(hons.) Kaveh Rahnema

SS 2014
Sheet 5
26.05.2013

Scientific Computing II

Krylov Methods

Exercise 6: Method of Steepest Descent

In this exercise, we want to derive and understand the method of steepest descent to solve the system of linear equations $A \cdot x = b$.

Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, $b \in \mathbb{R}^n$, and $f(x) := \frac{1}{2} x^T \cdot Ax - b^T \cdot x$.

(a) Describe the idea of and derive the steepest descent method.

(b) Prove that $\bigtriangledown f(x) = Ax - b$.

(c) Compute the first 3 iterations of the steepest descent method to minimize

$$f(x) := \frac{1}{2} x^T A x - b^T x \text{ for } x \in \mathbb{R}$$

with $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d^{(1)} = - \bigtriangledown f(x^{(1)} = b - Ax$.
What do you observe?

(d) Compute the exact solution of the system in (c) and compute the error after each iteration. By which factor is it reduced in each iteration?

(e) Write down the pseudo code for one iteration of the steepest descent method solving the two-dimensional Poisson equation on the unit interval with homogeneous Dirichlet boundary conditions:

$$\begin{aligned} \Delta u &= 0 \text{ in }]0,1[^2, \\ u &= 0 \text{ at } \partial]0,1[^2. \end{aligned}$$

The Laplacian is discretized in a regular grid by the 5-point-stencil

$$\Delta u(i \cdot h, j \cdot h) \approx \frac{u_{i-1,j} + u_{i,j-1} - 4u_{i,j} + u_{i+1,j} + u_{i,j+1}}{h^2}.$$

Use elementwise notation (i.e. not matrix-vector notation)!

(f) Give the cost per iteration of the steepest descent in the form $O(N^p)$ for general systems of linear equations! In addition, consider the one-dimensional Poisson equation with homogeneous Dirichlet conditions (cf. sheet 1).

(g) Consider the one-dimensional Poisson equation. Use the asymptotic convergence rate $\rho = \frac{\kappa-1}{\kappa+1}$ with $\kappa = \frac{\lambda_{max}}{\lambda_{min}}$ of the steepest descent method to derive the cost of the method if used as an iterative solver for this problem.
For the Poisson problem as discretized on sheet 1, $\lambda_{max} = 1 + \cos(\pi h)$ and $\lambda_{min} = 1 - \cos(\pi h)$.

```
for j=1:N do
  r_i^j = b_i^j;
  for k=1:N do
    r_i^j = r_i^j - A(j,k) * x_i^k;
  end
end
for j=1:N do
  a = a + r_i^j * r_i^j;
  for k=1:N do
    tmp^j = A(j,k) * r_i^k;
  end
  b = b + r_i^j * tmp^j;
end
alpha = a / b;

for j=1:N do
  x_i+1^j = x_i^j + alpha * r_i^j
end
```

Exercise 7: Conjugate Gradients Method

(a) Describe the idea of and derive the Conjugate Gradients method.

(b) Solve the system from 6 c) with CG. What do you observe?

(c) Give the cost per iteration of CG in the form $O(N^p)$for general systems of linear equations! In addition, consider the one-dimensional Poisson equation with homogeneous Dirichlet conditions (cf. sheet 1).

(d) Give the overall costs of CG if used as a direct solver! Compare to the results for Gaussian elimination.

(e) Consider the one-dimensional Poisson equation. Use the asymptotic convergence rate of CG $\rho = \frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$ to derive the cost of the method if used as an iterative solver. For the Poisson problem discretized as on sheet 1, $\lambda_{max} = 1 + \cos(\pi h)$ and $\lambda_{min} = 1 - \cos(\pi h)$.