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A V-cycle Multigrid for multilevel matrix algebras: proof of optimality ⋆

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Abstract We analyze the convergence rate of a multigrid method for multilevel linear systems whose coefficient matrices are generated by a real and nonnegative multivariate polynomial f and belong to multilevel matrix algebras like circulant, tau, Hartley, or are of Toeplitz type.

In the case of matrix algebra linear systems, we prove that the convergence rate is independent of the system dimension even in presence of asymptotical ill-conditioning (this happens iff f takes the zero value). More precisely, if the d-level coefficient matrix has partial dimension n_r at level r, with $r = 1, \ldots, d$, then the size of the system is $N(\boldsymbol{n}) = \prod_{r=1}^{d} n_r$, $\boldsymbol{n} = (n_1, \ldots, n_d)$, and $O(N(\boldsymbol{n}))$ operations are required by the considered V-cycle Multigrid in order to compute the solution within a fixed accuracy. Since the total arithmetic cost is asymptotically equivalent to the one of a matrix-vector product, the proposed method is optimal. Some numerical experiments concerning linear systems arising in 2D and 3D applications are considered and discussed.

Key words Matrix algebra – Toeplitz class – multilevel matrices – algebraic multigrid

Mathematics Subject Classification (1991): 65N55, 65F10, 65F15

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1 Introduction

Multigrid methods for tau and Toeplitz matrices were firstly introduced in [13] and then applied to two-level Toeplitz matrices in [14]. These early works led to several generalizations, other matrix algebras in [22,9], different projecting strategies for Toeplitz matrices in [16,7,10], and to further developments in the theoretical analysis on the convergence rate [7,10, 20,1]. In [1] it was proved that in the one-dimensional case the Algebraic Multi-Grid (AMG) firstly introduced in [13] for tau and Toeplitz matrices and in [22] for circulant matrices generated by nonnegative polynomial functions, under slightly stronger conditions, is optimal when using the V-cycle and only one iteration of relaxed Richardson as post-smoother. The optimality is in the sense of Axelsson and Neytcheva [2], i.e., the problem of solving a linear system with coefficient matrix $A_{\boldsymbol{n}}$ is asymptotically of the same cost as the direct problem of multiplying $A_{\boldsymbol{n}}$ by a vector. These slightly stronger conditions are translated in the choice of a projector which results more powerful than the previous proposals when the coefficient matrices possess a generating function with zeros of order greater than two.

In this paper we extend to the multidimensional case the analysis proposed in [1]. Therefore we prove that, under slightly stronger conditions, the AMG proposed in [14,20,22] for matrices that belong to multilevel circulant, tau or Hartley algebras and that are generated by nonnegative multivariate polynomial functions is optimal when using the V-cycle and at least one (pre or post) smoothing iteration of relaxed Richardson (a similar analysis can be done for other stationary methods). We will show that the total cost of the considered AMG is $O(N(\boldsymbol{n}))$ arithmetic operations, since:

1. all the matrices appearing in the AMG have a number of non-zero diagonals independent by $\boldsymbol{n}$ and they can be computed with a number of operations proportional to $\log(N(\boldsymbol{n}))$,
2. each iteration requires the same computational cost of the matrix-vector product, i.e. $O(N(\boldsymbol{n}))$ arithmetic operations,
3. the number of iterations required for the convergence is bounded by a constant which does not depend on $\boldsymbol{n}$.

The last point means that the convergence rate is independent of $N(\boldsymbol{n})$ and its proof in the multidimensional case is the main contribution of this paper. Furthermore, the matrices at each multigrid level belong to the same algebra and then the recursive V-cycle procedure is well defined.

In the case of the considered matrix algebras the cost by direct methods using fast transforms is $O(N(\boldsymbol{n}) \log N(\boldsymbol{n}))$ operations, while an optimal technique would require just $O(N(\boldsymbol{n}))$ operations. This kind of matrix algebra linear systems is widely encountered when preconditioning more complicated problems (dense multilevel Toeplitz systems, discretization of

multidimensional differential and/or integral equations, etc. [8,19]) or directly arises in some image restoration problems with shift-invariant kernel and suitable boundary conditions (see [15,21]).

The tau algebra is also the key to extend the proposed AMG to multilevel Toeplitz matrices. In this direction there are several proposals [16,20, 7,10,1]. The proposal in [16] does not follow a Galerkin strategy and hence there are not many useful tools for a theoretical analysis, while the proposal in [1] extends the one in [20] preserving more information at each recursion level when the generating function has a zero of arbitrary finite order. In [20] the optimality is proved in the Two-Grid case, while in [7,10] the level independency in the one and twodimensional case is also proved with generating function having zeros of order at most two. The latter implies the optimality using the W-cycle, but, as shown in [1], it is not enough for the optimality of the V-cycle. We emphasize that for multilevel Toeplitz matrices with nonnegative generating functions having a zero of order at most two, all the generalizations described in [16,20,7,10,1] of the original idea contained in [13,14] define exactly the same multigrid procedure. Furthermore for generating functions having zeros of order greater than two there are no results on the optimality of the cited proposals. We recall that in [20] the level independency is implicitly proved but not explicitly stated for a zero of arbitrary order. However the experimentation presented here and in [1] confirms numerically an optimal behavior of our proposal also for zeros of order greater than two already in the V-cycle case. In this case the fast direct techniques require a computational cost of $O([N(\boldsymbol{n})]^{\frac{3d-1}{d}})$ [17] and need further stabilization tricks, while the most popular preconditioning strategies can be far from being optimal [23].

With respect to [1,20], in this paper we provide a more precise analysis of the computational cost of one AMG iteration and of the choice of the optimal smoothers's relaxation parameter (it is the first time that a complete analysis is considered also for the pre-smoother and an iteration number greater than one), especially we extend to the multidimensional case the proof of optimality presented in [1]. The latter study leads to the choice of a more powerful projector with respect to [20] and, moreover, to derive it from [1] is not a trivial task. In addition we remark that generally it is not true that if we have optimality for a given iterative solver in the one-dimensional case, then the same property transfers to the multidimensional case. A notable example is the preconditioning of multilevel Toeplitz systems using multilevel algebras like the circulant algebra: indeed many optimal preconditioners can be found in the one dimensional case while in the multidimensional case this has been shown to be theoretically impossible, see [23]. On the other hand, by using multilevel band Toeplitz preconditioners (see e.g. [19]), it is possible to reduce the computation with

dense Toeplitz systems to the case of Toeplitz linear systems whose coefficient matrices are generated by nonnegative polynomials. Therefore it is of special interest to be able to solve in an optimal way these preconditioned systems and the latter can be performed with the proposed AMG. We expect that the theoretical tools introduced in this paper for the multilevel matrix algebra case can be employed for proving the AMG optimality in the multilevel Toeplitz context as well. However, to consider the algebra case instead of the Toeplitz case is the same simplification that is done in the classical Local Fourier Analysis for the geometric multigrid (see [6]). As an example some discretizations of PDEs with periodic boundary conditions lead to circulant matrices while Dirichlet boundary conditions lead to Toeplitz matrices.

The paper is organized as follow. In §2 we describe the multilevel circulant, tau and Hartley algebras and the multilevel Toeplitz matrices emphasizing their main common properties. In §3 we describe the V-cycle AMG procedure. In §4 we analyze three constraints that the AMG must satisfy in order to obtain an optimal method. In §5 we prove the convergence and optimality property of our AMG. §6 contains a wide numerical experimentation arising in 2D and 3D applications that confirms our theoretical analysis. Finally, §7 is devoted to concluding remarks.

2 Multilevel algebra and Toeplitz matrices

In this paper we will consider a multigrid method to solve linear systems whose matrices belong to multilevel circulant, tau and Hartley algebras. We will provide an uniform approach that in fact can be extended to other matrix algebras (for DCTIII see [9]).

Let $d, n \in \mathbb{N} \setminus \{0\}$, $\mathcal{F}_d = \{f : \mathbb{R}^d \to \mathbb{R}\}$ and let $\mathrm{Diag}(\mathbf{z})$ be the diagonal matrix with principal diagonal equal to $\mathbf{z} \in \mathbb{R}^n$. To any unitary matrix Q_n (i.e. $Q_n^{-1} = Q_n^H$) we can associate the Hermitian algebra $\mathcal{G}(Q_n) = \{Q_n \cdot \mathrm{Diag}(\mathbf{z}) \cdot Q_n^H \mid \mathbf{z} \in \mathbb{R}^n\}$ and hence the map $\mathcal{A}_n$ defined by

$$\begin{aligned} \mathcal{A}_n : \mathcal{F}_1 &\longrightarrow \mathcal{G}(Q_n) \\ f &\longrightarrow Q_n \cdot \mathrm{Diag}\, f(\mathbf{w}^{[n]}) \cdot Q_n^H \end{aligned}$$

is an algebra homomorphism where $\mathbf{w}^{[n]}$ is a fixed vector of $\mathbb{R}^n$ and $f(\mathbf{w}^{[n]})$ denotes the vector with components $f(w_i^{[n]})$. As a consequence $\mathbf{u}_i^{[n]} = Q_n \mathbf{e}_i$ is an unitary eigenvector of $\mathcal{A}_n(f)$ related to the eigenvalue $f(w_i^{[n]})$. The circulant matrix $\mathcal{C}_n(f)$, the tau matrix $\tau_n(f)$ and the Hartley matrix $\mathcal{H}_n(f)$ with $f \in \mathcal{F}_1$ can be written as $\mathcal{A}_n(f)$, $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}\}$, by means of the objects Q_n and $\mathbf{w}^{[n]}$ defined in Table 1.

	$\mathcal{A}$	$\mathcal{I}_n$	$\mathbf{w}^{[n]}$	Q_n
Circulants	$\mathcal{C}$	$0,\dots,n-1$	$w_i^{[n]} = \frac{2\pi i}{n}$	$F_n = \frac{1}{\sqrt{n}}\left[\mathrm{e}^{-\mathbf{i}jw_i^{[n]}}\right]_{i,j\in\mathcal{I}_n}$
Hartley	$\mathcal{H}$	$0,\dots,n-1$	$w_i^{[n]} = \frac{2\pi i}{n}$	$H_n = \mathrm{Re}(F_n) + \mathrm{Im}(F_n)$
Tau	τ	$1,\dots,n$	$w_i^{[n]} = \frac{\pi i}{n+1}$	$S_n = \sqrt{\frac{2}{n+1}}\left[\sin\left(jw_i^{[n]}\right)\right]_{i,j\in\mathcal{I}_n}$

Table 1. Basics of trigonometric algebras: index range, sampling points, eigenvectors.

In [1] we proposed an AMG (improving the one studied in [13]) to solve the linear system $\mathcal{A}_n(f)\,\mathbf{x} = \mathbf{b}$ with total arithmetic cost linear in n under the assumption that f is a trigonometric polynomial that vanishes in zero and is positive in the open interval $(0, 2\pi)$ (we required f even in the tau case): then it was shown how to extend the result to the case where the unique root is not at zero and in the case where f has more than one root. In this paper we extend this analysis to the multilevel case improving the previous proposals and the theoretical results in [14, 20, 22].

A d-level matrix $A_{\boldsymbol{n}}$ of partial dimension $\boldsymbol{n} = (n_1, n_2, \dots, n_d) \in (\mathbb{N}\setminus\{0\})^d$ can be described (see [26]) as a $n_1 \times n_1$ block matrix whose elements are $n_2 \times n_2$ block matrices and so on with d nesting levels; its true dimension is $N(\boldsymbol{n}) = \prod_{r=1}^{d} n_r$. We will refer to the elements of such matrix by using a couple $(\boldsymbol{i}, \boldsymbol{j})$ of d-indices: $A_{\boldsymbol{n}} = [a_{\boldsymbol{i},\boldsymbol{j}}]$, and the selected element $a_{\boldsymbol{i},\boldsymbol{j}}$ is the one in position (i_r, j_r) at the r-th level, for every $r = 1, \dots, d$. Circulant, tau and Hartley d-level matrix algebras of partial dimension $\boldsymbol{n}$ can be defined as the matrix algebra $\mathcal{G}(Q_{\boldsymbol{n}})$ associated with the transform $Q_{\boldsymbol{n}} = Q_{n_1} \otimes \cdots \otimes Q_{n_d}$. All Q_{n_i} have to be selected in the same row of Table 1: it is possible to deal with mixed structures and the corresponding multigrid analysis is straightforward (see e.g. [22]), but we will not emphasize this point hereafter. Of course we can associate multilevel matrices $\mathcal{C}_{\boldsymbol{n}}(f)$, $\tau_{\boldsymbol{n}}(f)$ and $\mathcal{H}_{\boldsymbol{n}}(f)$ to each multivariate function $f \in \mathcal{F}_d$, thus we extend the map $\mathcal{A}_n$ to $\mathcal{A}_{\boldsymbol{n}}$ as follows

$$\begin{aligned} \mathcal{A}_{\boldsymbol{n}} : \mathcal{F}_d &\longrightarrow \mathcal{G}(Q_{\boldsymbol{n}}) \\ f &\longrightarrow Q_{\boldsymbol{n}} \cdot \mathrm{Diag}\, f(\mathbf{w}^{[\boldsymbol{n}]}) \cdot Q_{\boldsymbol{n}}^H \end{aligned}$$

where the sampling point multilevel vector $\mathbf{w}^{[\boldsymbol{n}]} \in \mathbb{R}^{N(\boldsymbol{n})}$ is defined as $w_{\boldsymbol{i}}^{[\boldsymbol{n}]} = (w_{i_1}^{[n_1]}, \dots, w_{i_d}^{[n_d]})$, with $\boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}} := \mathcal{I}_{n_1} \times \cdots \times \mathcal{I}_{n_d}$. It follows that $\mathbf{u}_{\boldsymbol{i}}^{[\boldsymbol{n}]} = \mathbf{u}_{i_1}^{[n_1]} \otimes \cdots \otimes \mathbf{u}_{i_d}^{[n_d]}$ is an eigenvector related to the sampling in $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$. We note that $\mathcal{A}_{\boldsymbol{n}}$ is an algebra homomorphism as well, thus in particular $\mathcal{A}_{\boldsymbol{n}}(f)\mathcal{A}_{\boldsymbol{n}}(g) = \mathcal{A}_{\boldsymbol{n}}(fg)$ holds.

In this article we are interested in linear systems $\mathcal{A}_{\boldsymbol{n}}(f)\,\mathbf{x} = \mathbf{b}$ with $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}, T\}$, f being a nonnegative multivariate trigonometric polynomial. Once again, we require f to be even (with respect to each variable) in the tau case. Here $T_{\boldsymbol{n}}(f)$ is the Toeplitz d-level matrix of partial dimension $\boldsymbol{n}$ defined as

$$T_{\boldsymbol{n}}(f) = \sum_{|\boldsymbol{j}| \leqslant \boldsymbol{n}-\boldsymbol{e}} a_{\boldsymbol{j}} J_{\boldsymbol{n}}^{[\boldsymbol{j}]} = \sum_{|j_1|<n_1} \cdots \sum_{|j_d|<n_d} a_{(j_1,\ldots,j_d)} J_{n_1}^{[j_1]} \otimes \cdots \otimes J_{n_d}^{[j_d]}$$

($\boldsymbol{e} = (1,\ldots,1) \in \mathbb{N}^d$) by means of the Fourier coefficients of f

$$a_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\mathbf{x}) \mathrm{e}^{-\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \mathrm{d}\mathbf{x}, \qquad \mathbf{i}^2 = -1, \qquad \boldsymbol{k} \in \mathbb{Z}^d. \quad (1)$$

Here $J_n^{[j]} \in \mathbb{R}^{n\times n}$ is the matrix whose (s,t)-th entry equals 1 if $s - t = j$ and is 0 elsewhere.

We assume $f \in \mathbb{R}_{\boldsymbol{z}}$ with $\boldsymbol{z} \in (\mathbb{N} \setminus \{0\})^d$, where $\mathbb{R}_{\boldsymbol{\eta}}$, $\boldsymbol{\eta} \in \mathbb{N}^d$, is the set of d-variate real-valued trigonometric polynomials with degree up to $\boldsymbol{\eta}$:

$$\mathbb{R}_{\boldsymbol{\eta}}[\mathbf{x}] = \left\{ \sum_{|\boldsymbol{k}| \leqslant \boldsymbol{\eta}} a_{\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \ \text{ s.t. } \ a_{-\boldsymbol{k}} = \overline{a_{\boldsymbol{k}}} \in \mathbb{C} \right\}.$$

In this case it is known (see [26,3,4]) that all the matrices $\mathcal{A}_{\boldsymbol{n}}(f)$ are hermitian, banded (in the way induced from the considered structure) and semi-positive definite if $f \geqslant 0$. Moreover $\mathcal{A}_{\boldsymbol{n}}(f)$ is ill-conditioned whenever f takes the zero value; it is singular if $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}\}$ and f vanishes in a grid point $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$.

If $f \geqslant 0$ vanishes in the grid point $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$ then it is usually replaced by the positive function

$$f_+ = f + \min_{\|\boldsymbol{j}\|_\infty = 1} f\big(w_{\boldsymbol{i}+\boldsymbol{j}}^{[\boldsymbol{n}]}\big) \cdot \chi_{w_{\boldsymbol{i}}^{[\boldsymbol{n}]} + 2\pi\mathbb{Z}^d}, \quad (2)$$

here χ_S is the characteristic function of the set S, thus $\chi_S(\mathbf{x})$ is 1 if $\mathbf{x} \in S$ and is 0 if $\mathbf{x} \notin S$. Therefore $\mathcal{A}_{\boldsymbol{n}}(f)$ is consequently replaced by

$$\mathcal{A}_{\boldsymbol{n}}(f_+) = \mathcal{A}_{\boldsymbol{n}}(f) + \left(\min_{\|\boldsymbol{j}\|_\infty = 1} f\big(w_{\boldsymbol{i}+\boldsymbol{j}}^{[\boldsymbol{n}]}\big) \right) \cdot \mathbf{u}_{\boldsymbol{i}}^{[\boldsymbol{n}]} \Big(\mathbf{u}_{\boldsymbol{i}}^{[\boldsymbol{n}]}\Big)^H,$$

where, in the preconditioning literature, the rank-1 additional term is known as Strang correction [25].

3 Algebraic MultiGrid

Let $A \in \mathbb{C}^{N\times N}$ be an Hermitian positive definite matrix, $\mathbf{b} \in \mathbb{C}^N$, m integer with $0 < m < N$. Fix integers $N_0 = N > N_1 > N_2 > \cdots > N_m > 0$, take $R_i \in \mathbb{C}^{N_{i+1}\times N_i}$ full-rank matrices and consider two classes $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ of iterative methods for N_i-dimensional linear systems, $i = 0, \ldots, m-1$. The related AMG in the V-cycle version produces the sequence $\{\mathbf{x}^{(k)}\}_{k\in\mathbb{N}} \subset \mathbb{C}^N$ according to the rule $\mathbf{x}^{(k+1)} = \mathcal{AMG}(0, \mathbf{x}^{(k)}, \mathbf{b})$, with $\mathcal{AMG}$ recursively defined as follows (where $A_0 = A$, $\mathbf{b}_0 = \mathbf{b}$):

$$\mathbf{x}_i^{(\text{out})} := \mathcal{AMG}(\, i, \mathbf{x}_i^{(\text{in})}, \mathbf{b}_i \,)$$

$$\begin{array}{lll}
\texttt{If } (i = m) \texttt{ Then Solve}\,(A_m \mathbf{x}_m^{(\text{out})} = \mathbf{b}_m) & & \\
\qquad\qquad \texttt{Else} & \mathbf{1} & \mathbf{x}_i^{(\text{pre})} := \mathcal{S}_i^{\nu_i}\left(\mathbf{x}_i^{(\text{in})}\right) \\
& \mathbf{2} & \mathbf{r}_i \quad := \mathbf{b}_i - A_i \mathbf{x}_i^{(\text{pre})} \\
& \mathbf{3} & \mathbf{b}_{i+1} := R_i \mathbf{r}_i \\
& \mathbf{4} & A_{i+1} := R_i A_i (R_i)^H \\
& \mathbf{5} & \mathbf{x}_{i+1}^{(\text{out})} := \mathcal{AMG}(i+1, \mathbf{0}_{N_{i+1}}, \mathbf{b}_{i+1}) \\
& \mathbf{6} & \mathbf{x}_i^{(\text{int})} := \mathbf{x}_i^{(\text{pre})} + R_i^H \mathbf{x}_{i+1}^{(\text{out})} \\
& \mathbf{7} & \mathbf{x}_i^{(\text{out})} := \widetilde{\mathcal{S}}_i^{\vartheta_i}\left(\mathbf{x}_i^{(\text{int})}\right)
\end{array} \tag{3}$$

Step **1** performs some (ν_i) iterations of a "pre-smoother"; step **2** calculates the residue of presmoother approximation; steps **3**, **4**, **5** and **6** define the *recursive coarse grid correction* by restriction (**3**) of the residue, coarse grid correction (**4**, **5**) and interpolation (**6**), while step **7** performs some (ϑ_i) iterations of a "post-smoother".

The restrictors R_i have to be full-rank, thus all A_i are nonsingular, hermitian and positive definite. Most of the times smoothers are one-point methods:

$$\begin{cases} \mathcal{S}_i(\mathbf{x}) = S_i\,\mathbf{x} + (I_{N_i} - S_i)A_i^{-1}\mathbf{b}_i \\ \widetilde{\mathcal{S}}_i(\mathbf{x}) = \widetilde{S}_i\,\mathbf{x} + (I_{N_i} - \widetilde{S}_i)A_i^{-1}\mathbf{b}_i \end{cases}, \quad \mathbf{x} \in \mathbb{C}^{N_i}, \quad i = 0, \ldots, m-1. \tag{4}$$

Steps **2**-**6** allow us to define on each level i the *exact coarse grid correction operator* :

$$CGC_i = I_{N_i} - {R_i}^H A_{i+1}^{-1} R_i\, A_i, \qquad i = 0, \ldots, m-1. \tag{5}$$

Under these assumptions, it is possible to prove that the AMG is a one-point method and its linear part AMG_0 is recursively defined as

$$\begin{cases} AMG_m = O_{N_m \times N_m} \\ AMG_i \ = \widetilde{S}_i^{\vartheta_i} \cdot \Big[I_{N_i} - {R_i}^H \big(I_{N_{i+1}} - AMG_{i+1} \big) A_{i+1}^{-1} R_i \, A_i \Big] \cdot S_i^{\nu_i}, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = m-1, \ldots, 0. \end{cases} \tag{6}$$

This shows that, unless we are in the two-grid case, by swapping the order of smoothers (or else by applying both before or after the recursive coarse grid correction) we affect the spectra of AMG_0.

4 The AMG for matrix algebras

To reach convergence and optimality, and what is more to write a good algorithm, we have to answer three requests of different nature: algebraic, computational, and convergence-optimality.

The *algebraic* requirement (§4.1) is the following: every matrix A_i generated from step **4** of AMG algorithm (3) has to be in the same algebra $\mathcal{G}$ of A_0 and hence

$$A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i) \in \mathcal{G}(Q_{\boldsymbol{n}_i}) \tag{7}$$

has to hold, f_i being a suitable function (in the following $f = f_0$, $\boldsymbol{z} = \boldsymbol{z}_0$ and $\boldsymbol{n} = \boldsymbol{n}_0$ by choice) and $\boldsymbol{n}_i$ a suitable multiindex. This means that the matrices A_i generated from step **4** of algorithm (3) have all to be circulant, or all tau or all Hartley, each one of the right partial order. It is obvious that the algebraic requirement does not imply convergence and optimality by itself, but it is necessary to define a recursive technique and also to obtain a good method: since the coarse grid matrix has to approximate the fine grid matrix, if they are of the same matrix algebra type (e.g. circulant) then the approximation would likely be better.

The *computational* requirement (§4.2) is related to optimality: the computational cost on each iteration has to be as low as possible, i.e.

$$O(N_0) = O(N(\boldsymbol{n}_0))$$

since we deal with banded matrices. This is reached if the following three conditions are guaranteed:

1. $\{R_i\}_{i=0}^{m-1}$ and $\{A_i\}_{i=1}^{m}$ can be (pre)computed with cost at most $O(N_0)$;
2. the products $A_i \, \mathbf{x}$, $R_i \, \mathbf{r}$ and ${R_i}^H \mathbf{y}$ (steps **2**, **3** and **6**) and smoothers (steps **1** and **7**) have linear cost with respect to the dimension $N(\boldsymbol{n}_i)$;
3. the cost of solving $A_m \mathbf{x}_m^{(\text{out})} = \mathbf{b}_m$ is at most $O(N_0)$.

	Circulant & Hartley algebra	τ algebra
N_0	2^t	2^t-1
N_i	$\frac{N_{i-1}}{2}=2^{t-i}$	$\frac{N_{i-1}-1}{2}=2^{t-i}-1$
K_{N_i}	$\begin{bmatrix} 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{bmatrix}_{N_{i+1}\times N_i}$	$\begin{bmatrix} 0 & 1 & 0 & & \\ & 0 & 1 & 0 & \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 1 & 0 \end{bmatrix}_{N_{i+1}\times N_i}$
$K_{N_i}Q_i$	$\left[\, Q_{N_{i+1}} \middle\vert Q_{N_{i+1}} \,\right]$	$\left[\, Q_{N_{i+1}} \middle\vert 0_{N_{i+1}} \middle\vert M_n Q_{N_{i+1}} \,\right]$
R_i	$K_{N_i}\ \mathcal{A}_{N_i}(p_i)$	

Table 2. Scalar case: dimensions, cutting operators and relations ($M_n=\mathrm{Diag}_{r=1}^{n}(-1)^r$).

The *convergence-optimality* requirement (§4.3) is the following: the error reduction on each iteration has to be smaller than one (convergence) and also uniformly bounded (optimality), with respect to the dimension of the problem, by a constant smaller than one and independent of N_0 and m. It follows that this constant will depend only on the generating function:

$$\rho(AMG_0)\leqslant \mathrm{const}(f_0)<1,$$

$\rho(M)$ being the spectral radius of M. Convergence and optimality are the core of this article and will be proved in §5.

4.1 Algebraic requirement

Here we describe how to satisfy the algebraic requirement (7). We simply give the multilevel version of the arguments defined in the one-level case in [1] according to Table 2. In [1] we fixed $N_i=2^{t-i}$ for circulants and Hartley and $N_i=2^{t-i}-1$ for tau (t is an integer number) and we choose as projector (restrictor) R_i the product between a cutting matrix K_{N_i} (defined in Table 2) and a matrix $\mathcal{A}_{N_i}(p_i)$ in the algebra $\mathcal{G}(Q_{N_i})$, each $p_i\in\mathcal{F}_1$ being a trigonometric polynomial. By means of cutting relations we obtained $A_i=\mathcal{A}_{N_i}(f_i)$, being $\{f_i\}_{i=0}^{m}\subset\mathcal{F}_1$ defined by $f_{i+1}=\Psi_1(p_i^2 f_i)$, with $\Psi_1:\mathcal{F}_1\longrightarrow\mathcal{F}_1$ defined as follows [13]:

$$\Psi_1\big[g(x)\big]=\frac{1}{2}\left[g\Big(\frac{x}{2}\Big)+g\Big(\frac{x}{2}+\pi\Big)\right].$$

Now we deal with the d-level case, starting with $A_0=\mathcal{A}_{\boldsymbol{n}_0}(f_0)$ whose partial order is $\boldsymbol{n}_0=2^t\boldsymbol{e}\in\mathbb{N}^d$ for circulants and Hartley and $\boldsymbol{n}_0=(2^t-1)\boldsymbol{e}$ for tau, where $\boldsymbol{e}=(1,\ldots,1)\in\mathbb{N}^d$ and t is still a positive integer. We essentially halve each partial order on each level, by defining $\boldsymbol{n}_i=2^{t-i}\boldsymbol{e}$ for circulants and Hartley and $\boldsymbol{n}_i=(2^{t-i}-1)\boldsymbol{e}$ for tau. As projector

R_i we choose again a product, between a d-level cutting matrix $K_{\boldsymbol{n}_i} = K_{(n_i)_1} \otimes \cdots \otimes K_{(n_i)_d}$ and a matrix $\mathcal{A}_{\boldsymbol{n}_i}(p_i)$ in the d-level algebra $\mathcal{G}(Q_{\boldsymbol{n}_i})$, $p_i \in \mathcal{F}_d$ (see Table 3).

	Circulant & Hartley algebra	τ algebra
$\boldsymbol{n}_0$	$2^t\boldsymbol{e}$	$(2^t-1)\boldsymbol{e}$
$\boldsymbol{n}_i$	$\frac{\boldsymbol{n}_{i-1}}{2} = 2^{t-i}\boldsymbol{e}$	$\frac{\boldsymbol{n}_{i-1}-\boldsymbol{e}}{2} = (2^{t-i}-1)\boldsymbol{e}$
$m_{\max}$	t	$t-1$
$K_{\boldsymbol{n}_i}$	$K_{(n_i)_1} \otimes \cdots \otimes K_{(n_i)_d}$	
R_i	$K_{\boldsymbol{n}_i}\ \mathcal{A}_{\boldsymbol{n}_i}(p_i)$	

Table 3. Multilevel case ($d > 1$): dimensions and cutting operators ($\boldsymbol{e} = (1, \ldots, 1)$).

These choices preserve a d-level structure in each A_i, because of the inductive step $A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i) \ \Rightarrow\ A_{i+1} = \mathcal{A}_{\boldsymbol{n}_{i+1}}(f_{i+1})$:

$$\begin{aligned} A_{i+1} &= R_i\ A_i\ (R_i)^H \\ &= K_{\boldsymbol{n}_i}\ \mathcal{A}_{\boldsymbol{n}_i}(p_i^2 f_i)\ (K_{\boldsymbol{n}_i})^H \\ &= K_{\boldsymbol{n}_i} Q_{\boldsymbol{n}_i}\ \mathrm{Diag}\left((p_i^2 f_i)(\mathbf{w}^{[\boldsymbol{n}_i]})\right)\ Q_{\boldsymbol{n}_i}^H K_{\boldsymbol{n}_i}^T \\ &= Q_{\boldsymbol{n}_{i+1}}\ \mathrm{Diag}\left(\left[\Psi_d(p_i^2 f_i)\right](\mathbf{w}^{[\boldsymbol{n}_{i+1}]})\right)\ Q_{\boldsymbol{n}_{i+1}}^H \end{aligned}$$

(see [14,22] for details on last equality, where we assume all functions to be even in the tau case). This leads to the following

Proposition 1 (Algebraic requirement) *With notations of Table 3, let t, $m \in \mathbb{N}$ be such that $0 < m < t$ and let f_0, $p_i \in \mathcal{F}_d$ be 2π-periodic functions (even in tau case) for $i = 0, \ldots, m-1$. Define also $A_{i+1} = R_i\ A_i\ (R_i)^H$ for $i = 0, \ldots, m-1$. Then it holds $A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i)$, $i = 0, \ldots, m$, $\{f_i\}_{i=0}^m$ being defined as*

$$f_{i+1} = \Psi_d(p_i^2 f_i) \tag{8}$$

and $\Psi_d :\ \mathcal{F}_d \longrightarrow \mathcal{F}_d$ defined as

$$\Psi_d\big[g(\mathbf{x})\big] = \frac{1}{2^d} \sum_{\boldsymbol{s} \in \{0;1\}^d} g\Big(\frac{\mathbf{x}}{2} + \pi\boldsymbol{s}\Big). \tag{9}$$

Moreover the projector R_i is full-rank if $\Psi_d[p_i^2(\mathbf{x})] > 0$ holds for every $\mathbf{x}$.

4.2 Computational requirement

As we stated in §1, we are interested in linear systems generated by a polynomial $f_0 \in \mathbb{R}_{\boldsymbol{z_0}}$: this means that the first matrix of the sequence $\{A_i\}_{i=0}^m$ is structured (i.e. $A_0 = \mathcal{A}_{\boldsymbol{n}_0}(f_0) \in \mathcal{G}(Q_{\boldsymbol{n}_0})$) and sparse ($f_0 \in \mathbb{R}_{\boldsymbol{z}_0}$), while all A_i are still structured ($A_i = \mathcal{A}_{\boldsymbol{n}_i}(f)$ for §4.1), even if they can be dense. We assert that under the simple assumption that all the projector's generators p_i are polynomials, then all the matrices A_i have a number of non-zero diagonals lower than a constant independent by $\boldsymbol{n}$ and m. As a consequence, it is possible to guarantee that each iteration of the AMG (3) has a cost proportional to $N(\boldsymbol{n})$.

To show this result we have to analyze in detail how Ψ_d acts on polynomials: applying Ψ_d to a generic polynomial $\sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} a_{\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}$ we obtain

$$\begin{aligned}\Psi_d\left(\sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} a_{\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}\right) &= \sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} \left(\frac{1}{2^d} \sum_{\boldsymbol{s} \in \{0;1\}^d} \mathrm{e}^{\pi \mathbf{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle}\right) a_{\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \frac{\boldsymbol{k}}{2}|\mathbf{x}\rangle} \\ &= \sum_{\lceil \frac{\boldsymbol{c}_1}{2} \rceil \leqslant \boldsymbol{k} \leqslant \lfloor \frac{\boldsymbol{c}_2}{2} \rfloor} a_{2\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \end{aligned} \tag{10}$$

with componentwise floor and ceiling. The second equality follows from an orthogonality result[1]:

$$\sum_{\boldsymbol{s} \in \{0;1\}^d} \mathrm{e}^{\pi \mathbf{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle} = \begin{cases} 2^d & \text{if } k_r \equiv 0 \mod 2 \quad \forall r \in \{1,\dots,d\}, \\ 0 & \text{if } \exists\, \bar{r} \in \{1,\dots,d\} \text{ s.t. } k_{\bar{r}} \equiv 1 \mod 2. \end{cases}$$

In particular we get

$$\Psi_d(\mathbb{R}_{\boldsymbol{\eta}}[\mathbf{x}]) \subseteq \mathbb{R}_{\lfloor \frac{\boldsymbol{\eta}}{2} \rfloor}[\mathbf{x}]. \tag{11}$$

If a Strang correction is needed in the generating function f_0, then it could be useful (see §5.3) to have it also in the restrictor polynomials p_i. We observe that

$$(p + \lambda_1 \chi_{\{a\}})^2 (f + \lambda_2 \chi_{\{a\}}) = p^2 f + \lambda_3 \chi_{\{a\}} \tag{12}$$

with $\lambda_3 = \lambda_3(\lambda_1, \lambda_2, p(a), f(a))$ and Ψ_d is linear.
A simple result shows how the degree of $f_{i+1} = \Psi_d(g_i f_i)$ evolves when g_i are polynomials.

[1] It is straightforward if all k_r are even; if it exists $\bar{r}$ such that $k_{\bar{r}}$ is odd, it is enough to split $\{0;1\}^d = J_0 \cup J_1$, being $J_\theta = \left\{\boldsymbol{d} \in \{0;1\}^d \text{ s.t. } d_{\bar{r}} = \theta\right\}$. It holds $J_0 \simeq J_1 \simeq \{0,1\}^{d-1}$ and $\sum_{\boldsymbol{s} \in J_0} \mathrm{e}^{\pi \mathbf{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle} + \sum_{\boldsymbol{s} \in J_1} \mathrm{e}^{\pi \mathbf{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle} = 0$

Proposition 2 (Computational requirement) *Under the same assumptions of Proposition 1, let p_i be polynomials such that $p_i \in \mathbb{R}_{\boldsymbol{q}_i}$ and assume $f_0 \in \mathbb{R}_{\boldsymbol{z}_0}$. Then the following properties hold:*

1. *each f_i is a polynomial;*
2. *$f_i \in \mathbb{R}_{\boldsymbol{z}_i}$, being $\boldsymbol{z}_{i+1} \leqslant \boldsymbol{q}_i + \left\lfloor \frac{\boldsymbol{z}_i}{2} \right\rfloor$;*
3. *$\boldsymbol{z}_i \leqslant \max\{\boldsymbol{z}_0, 2\boldsymbol{q}_j \,:\, 0 \leqslant j < i\}$;*
4. *if $\boldsymbol{q}_i = \boldsymbol{q}$ for each i, then $\boldsymbol{z}_i \leqslant 2\boldsymbol{q}$ for i large enough (it depends on $\boldsymbol{z}_0 - 2\boldsymbol{q}$).*

Here multiindex inequalities and max*imum hold componentwise as usual.*

Proof Points *1* and *2* follow from (8) and (11).
Inequalities *3* can be showed by induction: $i = 0$ is clear; if $\boldsymbol{z}_i \leqslant 2\boldsymbol{q}_i$ then $\boldsymbol{z}_{i+1} \leqslant 2\boldsymbol{q}_i$, while if $\boldsymbol{z}_i \geqslant 2\boldsymbol{q}_i$ it holds $\boldsymbol{z}_{i+1} \leqslant \boldsymbol{z}_i \leqslant \max\{\boldsymbol{z}_0, 2\boldsymbol{q}_j \,:\, 0 \leqslant j < i\}$.
Inequality *4* follows from point *3* if $\boldsymbol{z}_0 \leqslant 2\boldsymbol{q}$, otherwise, like described in point *3*, $\{\boldsymbol{z}_i\}_{i\in\mathbb{N}}$ decreased strictly until for an index k holds $\boldsymbol{z}_k \leqslant 2\boldsymbol{q}$ and then $\boldsymbol{z}_{k+i} \leqslant 2\boldsymbol{q}$ for any $i \in N$. □

From Proposition 2 follows that if $p_i \in \mathbb{R}_{\boldsymbol{q}}$ for all i (we will show in §5.3 that this happen in our case) then the number of nonzero diagonals of the coefficient matrix at each multigrid recursion level is lower than $\prod_{r=1}^{d} 2\boldsymbol{q}_r + 1$. Therefore, it is easy to prove that, with a suitable choice of the smoother, one iteration of the algorithm $\mathcal{AMG}$ (3) is linear in N_0. This is done in the following lemma.

Lemma 1 *Under the same assumptions of Proposition 1 and*

1. *$p_i \in \mathbb{R}_{\boldsymbol{q}}$, for $i = 0, \ldots, m-1$,*
2. *pre and post-smoother are Richardson with $\nu_i + \vartheta_i \leqslant h \cdot (2^d - 1)^i$, where $h \geqslant 1$ is a constant,*

one iteration of the $\mathcal{AMG}$ in (3) *has a computation cost linear in N_0.*

The above lemma does not consider the cost $C_{\{A_i\}_{i=1}^m}$ of calculating the matrices $\{A_i\}_{i=1}^m$, i.e. of calculating the functions $\{f_i\}_{i=1}^m$. This can be done before the first iteration with logarithmic cost in N_0. Indeed, from Proposition 1 and from equation (10), it follows that we can get the coefficients of each f_{i+1} by computing the product $p_i^2 f_i$, where $p_i \in \mathbb{R}_{\boldsymbol{q}}$ and the relative $f_i \in \mathbb{R}_{\max\{\boldsymbol{z_0}, \boldsymbol{2q}\}}$ holds for each i. Since $\boldsymbol{q}$ and $\boldsymbol{z}_0$ do not depend on $\boldsymbol{n}_0$ but only on f_0, and we have to repeat this calculation $m-1$ times, it follows that there exists a constant $c(f_0)$ such that $C_{\{A_i\}_{i=1}^m} \leqslant c(f_0) \cdot m$ and m is less than $\log(N(\boldsymbol{n}_0))$.

Concluding, under the assumptions of §4.1 (Table 3) and of Lemma 1 ($p_i \in \mathbb{R}_{\boldsymbol{q}}$ and $\nu_i + \vartheta_i \leqslant h\,(2^d - 1)^i$), we know that each iteration of

$\mathcal{AMG}$ has linear cost, but it still remains to show the convergence and to check that the error reduction is constant with respect to the dimension $\boldsymbol{n}_0$. For this purpose we use a general result of Ruge and Stüben (see [18]), introduced in the next subsection, and we will show the validity of their hypotheses using linear algebra and functional tools (this is done in §5).

4.3 Convergence-optimality requirement

In the following, whenever X is an Hermitian positive definite matrix we define $\| \cdot \|_X = \|X^{1/2} \cdot \|_2$, being $\| \cdot \|_2$ the usual Euclidean norm on $\mathbb{C}^n$ or also the induced norm on $\mathbb{C}^{n\times n}$.

The proof of convergence for a V-cycle Multigrid is never trivial, since it mixes information from each grid in order to prove convergence. In a general framework, a typical convergence analysis is based on an estimate of the energy norm of the error transfer operator and this estimate is obtained from a factorization (see [5,28]) of the error transfer operator itself. Such approach is better understood with operator notation: if we define $H_0 = \mathbb{C}^{N_0}$, $H_1 = \text{Range}(R_0^H)$, $H_2 = \text{Range}(R_0^H R_1^H)$, $\ldots$, $H_m = \text{Range}(R_0^H R_1^H \ldots R_{m-1}^H)$ we get the sequence of nested spaces $H_0 \supset H_1 \cdots \supset H_m$ which is underlying the algorithm (3). We may also identify *two* class of projectors from H_0 onto H_i, $(i = 1, \ldots, m)$

- $R_{0i} = R_{i-1} \ldots R_1 \ldots R_0$ $(i = 1, \ldots, m)$, which is the one used in the algorithm (3)
- $P_i = R_{0i}^H A_i^{-1} R_{0i} A$, which is A-orthogonal.

If we would follow the approach of [5,28], the main task will be to prove this condition: for any $i = 1, \ldots, m$ and for any $\mathbf{x}_i \in H_i$ find γ_i such that

$$\|(P_i - P_{i+1})\mathbf{x}_i\|^2 \leqslant \gamma_i \|\mathbf{x}_i\|_A^2$$

(with $P_{m+1} = O$). Unfortunately it seems to be difficult to deal with the above condition, which can be rewritten as: for any $i = 1, \ldots, m$ and for any $\mathbf{y}_i \in \mathbb{C}^{N_i}$ find γ_i such that

$$\|R_{0i}^H C G C_i \mathbf{y}_i\|^2 \leqslant \gamma_i \|R_{0i}^H \mathbf{y}_i\|_A^2.$$

This inequality is more complicate than subsequente equation (15c); the condition stated above still needs to be investigated but hopefully will be the core of a future work.
To overcome this difficuly and to prove the Multigrid convergence we resorted to a different approach which has been proposed by Ruge and Stüben and is based on the following theorem.

Theorem 1 ([18]) *Let m, N be integers satisfying $0 < m < N$ and suppose that $A \in \mathbb{C}^{N\times N}$ is a positive definite Hermitian matrix and $\mathbf{b} \in \mathbb{C}^N$; given a sequence of $m+1$ positive integers $N = N_0 > N_1 > \cdots > N_m$, let $R_i \in \mathbb{C}^{N_{i+1}\times N_i}$ be full-rank matrices for each $i = 0, \ldots, m-1$. Define $A_0 = A$, $\mathbf{b}_0 = \mathbf{b}$ and choose two classes of iterative methods $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ as in* (4)*. If there exist two real positive numbers δ_{pre}, δ_{post} satisfying*

$$\|S_i^{\nu_i}\,\mathbf{x}\|_{A_i}^2 \leqslant \|\mathbf{x}\|_{A_i}^2 - \delta_{\text{pre}}\,\|CGC_i\,S_i^{\nu_i}\,\mathbf{x}\|_{A_i}^2 \qquad \forall\,\mathbf{x}\in\mathbb{C}^{N_i} \tag{13a}$$

and

$$\|\widetilde{S}_i^{\vartheta_i}\mathbf{x}\|_{A_i}^2 \leqslant \|\mathbf{x}\|_{A_i}^2 - \delta_{\text{post}}\,\|CGC_i\,\mathbf{x}\|_{A_i}^2 \qquad \forall\,\mathbf{x}\in\mathbb{C}^{N_i} \tag{13b}$$

both for every $i = 0, \ldots, m-1$, then it holds $\delta_{\text{post}} \leqslant 1$ and

$$\|AMG_0\|_A \leqslant \sqrt{\frac{1-\delta_{\text{post}}}{1+\delta_{\text{pre}}}} < 1. \tag{14}$$

Remark 1 From Theorem 1 the sequence $\{\mathbf{x}^{(k)}\}_{k\in\mathbb{N}}$ converges to the solution of $A\,\mathbf{x} = \mathbf{b}$ and besides when at least one between δ_{pre} and δ_{post} is independent of N and m, it converges with a constant error reduction not depending on N and m.

There are two point in [18] which might be misleading so we like to clarify them. In §4.3.1 of [18] at beginning the authors give partial a proof with only the hypotesis (13b), then they consider only the hypothesis (13a) and eventually they use both to prove Theorem 1. Later in §4.3.2 they suggest a way to split (13a) and (13b), namely

$$\begin{cases} \|S_i^{\nu_i}\,\mathbf{x}\|_{A_i}^2 & \leqslant \|\mathbf{x}\|_{A_i}^2 - \alpha\,\|S_i^{\nu_i}\,\mathbf{x}\|_{A_iD_i^{-1}A_i} \\ \|CGC_i\,\mathbf{x}\|_{A_i}^2 & \leqslant \gamma\,\|\mathbf{x}\|_{A_iD_i^{-1}A_i}^2 \\ \delta_{\text{pre}} = \alpha/\gamma & \end{cases} \tag{13.a-bis}$$

for (13a) and

$$\begin{cases} \|\widetilde{S}_i^{\vartheta_i}\,\mathbf{x}\|_{A_i}^2 & \leqslant \|\mathbf{x}\|_{A_i}^2 - \beta\,\|\mathbf{x}\|_{A_iD_i^{-1}A_i}^2 \\ \|CGC_i\,\mathbf{x}\|_{A_i}^2 & \leqslant \gamma\,\|\mathbf{x}\|_{A_iD_i^{-1}A_i}^2 \\ \delta_{\text{post}} = \beta/\gamma & \end{cases} \tag{13.b-bis}$$

and for (13b), where D_i is the diagonal part of A_i.
The first observation is that the $A_iD_i^{-1}A_i$-norm is not complusory: any other vector norm will work as well, provided that the same norm is used in the same block of braces. The A^2-norm is good for our purposes.
The second observation is about the coefficients α, β and γ. They can be

different when i changes, since the step from (13.a-bis) to (13a) and from (13.b-bis) to (13b) is purely algebraic and does not affect the proof of Theorem 1. This means that we can use the inequalities

$$\|S_i^{\nu_i}\,\mathbf{x}\|_{A_i}^2 \leqslant \|\mathbf{x}\|_{A_i}^2 - \alpha_i\,\|S_i^{\nu_i}\,\mathbf{x}\|_{A_i^2}^2 \quad (\alpha_i \geqslant 0), \quad \forall\,\mathbf{x}\in\mathbb{C}^{N_i}, \quad \text{(15a)}$$

$$\|\widetilde{S}_i^{\vartheta_i}\,\mathbf{x}\|_{A_i}^2 \leqslant \|\mathbf{x}\|_{A_i}^2 - \beta_i\,\|\mathbf{x}\|_{A_i^2}^2 \quad (\beta_i \geqslant 0), \quad \forall\,\mathbf{x}\in\mathbb{C}^{N_i}, \quad \text{(15b)}$$

$$\|CGC_i\,\mathbf{x}\|_{A_i}^2 \leqslant \gamma_i\,\|\mathbf{x}\|_{A_i^2}^2 \quad (\gamma_i > 0), \quad \forall\,\mathbf{x}\in\mathbb{C}^{N_i}. \quad \text{(15c)}$$

which are not weaker than (13), provided that it holds

$$\delta_{\text{pre}} = \min_{0\leqslant i<m}\frac{\alpha_i}{\gamma_i}, \qquad \delta_{\text{post}} = \min_{0\leqslant i<m}\frac{\beta_i}{\gamma_i}. \tag{16}$$

for every $i = 0,\ldots,m-1$.
We refer to (15a) as the *presmoothing property*, (15b) as the *postsmoothing property* and (15c) as the *approximation property* (see [18]). The approximation property depends exclusively on the choice of the projectors R_i but not on smoothers, whereas the smoothing properties are not related to R_i. The separate study of these properties allows us to cope with the more difficult part of the procedure, the verification of condition (15c), which involves the projectors but is independent of the smoothers.

We also notice that our condition (15c) is much stronger than (5.4) and (5.5) of [18], which are used in §4.5.3 of [18] to deal the Two-Grid convergence.

However, in order to fulfil conditions (13a) and (13b) with δ_{pre}, δ_{post} independent of $\boldsymbol{n}$ and m (which in turn imply the AMG optimal convergence by Theorem 1), we will show (see §5.3) that positive sequences $\{\alpha_i\}$, $\{\beta_i\}$ and $\{\gamma_i\}$ can be found such that the two ratios α_i/γ_i, β_i/γ_i converge to two positive constants if i goes to infinity. It follows that *the optimality is characterized by satisfaction of at least one of the two next* $\inf-\min$ *conditions:*

$$\inf_t \min_{0\leqslant i<m_{\max}(t)}\frac{\alpha_i}{\gamma_i} > 0, \qquad \inf_t \min_{0\leqslant i<m_{\max}(t)}\frac{\beta_i}{\gamma_i} > 0. \tag{17}$$

Summarizing all the results in this section:

Remark 2 Under assumptions of Table 3 and $p_i \in \mathbb{R}_{\boldsymbol{q}}$, $\nu_i+\vartheta_i \leqslant h\,(2^d-1)^i$, it is possible to demonstrate convergence and optimality for AMG algorithm (3) if all inequalities (15a,b,c) and at least one of (17) hold.

5 Proof of convergence and optimality

In this section we show how to ensure (15a,b,c) and especially (17). The smoothing properties (15a,b) will be discussed in §5.1 and proved in Proposition 3, the approximation property (15c) will be discussed in §5.2 and proved in Proposition 4, and finally in §5.3 we prove the optimality (i.e., the $\inf - \min$ condition (17)) in the concluding Theorem 2.

$Prop1$

5.1 Smoothing properties

In the following proposition we consider smoothers at a fixed recursion level and therefore, in order to simplify the notation, we do not use the grid index i. If X and Y are Hermitian matrices then $X \leqslant Y$ means that $Y - X$ is positive semidefinite.

Proposition 3 *Let $A = \mathcal{A}_{\boldsymbol{n}}(f)$ being $f \in \mathcal{F}_d$ nonnegative and not identically zero and let ω be a real number. If we define $S = I_{N(\boldsymbol{n})} - \omega A$, then*

$$\|S^\nu \mathbf{x}\|_A^2 \leqslant \|\mathbf{x}\|_A^2 - \alpha \|S^\nu \mathbf{x}\|_{A^2}^2, \qquad (\alpha \geqslant 0), \qquad \forall \mathbf{x} \in \mathbb{R}^{N(\boldsymbol{n})} \quad (18)$$

holds with $\nu \in \mathbb{N}$ if one of the following two is satisfied:

1. $0 \leqslant \omega \leqslant 1/\|f\|_\infty$ *and* $\alpha \leqslant 2\,\omega\,\nu$;
2. $1/\|f\|_\infty < \omega \leqslant 2/\|f\|_\infty$ *and*

$$\alpha \leqslant \min\left\{2\,\omega\,\nu,\ \frac{1}{\|f\|_\infty}\left[\frac{1}{\left(1-\omega\,\|f\|_\infty\right)^{2\nu}} - 1\right]\right\}.$$

Moreover if we define $\widetilde{S} = I_{N(\boldsymbol{n})} - \omega A$, then

$$\|\widetilde{S}^\vartheta \mathbf{x}\|_A^2 \leqslant \|\mathbf{x}\|_A^2 - \beta \|\mathbf{x}\|_{A^2}^2, \qquad (\beta \geqslant 0), \qquad \forall \mathbf{x} \in \mathbb{R}^{N(\boldsymbol{n})} \quad (19)$$

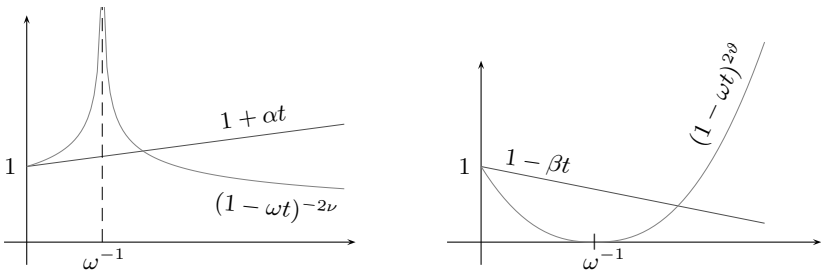
holds with $\vartheta \in \mathbb{N}$ if $0 \leqslant \omega \leqslant 2/\|f\|_\infty$ and

$$\beta \leqslant \frac{1 - \left(1 - \omega\,\|f\|_\infty\right)^{2\vartheta}}{\|f\|_\infty}$$

are satisfied.

Proof The scalar inequality (18) is nothing more than the Hermitian matrix inequality

$$(I_{N(\boldsymbol{n})} - \omega A)^\nu A (I_{N(\boldsymbol{n})} - \omega A)^\nu \leqslant A - \alpha (I_{N(\boldsymbol{n})} - \omega A)^\nu A^2 (I_{N(\boldsymbol{n})} - \omega A)^\nu$$



(a) Presmoother (b) Postsmoother

Figure 1. Functions related to smoother's inequalities.

and by $A = \mathcal{A}_{\boldsymbol{n}}(f) = Q_{\boldsymbol{n}}^H \operatorname{Diag}\big(f(\mathbf{w}^{[\boldsymbol{n}]})\big)\, Q_{\boldsymbol{n}}$ this is implied by a function inequality:

$$(1-\omega f)^{\nu} f\,(1-\omega f)^{\nu} \leqslant f - \alpha\,(1-\omega f)^{\nu} f^2 (1-\omega f)^{\nu},$$

and being $0 \leqslant f$ we rewrite it as $1+\alpha f \leqslant 1/(1-\omega f)^{2\nu}$, thus (18) is implied by

$$1+\alpha t \leqslant \frac{1}{(1-\omega t)^{2\nu}}, \qquad 0 < t \leqslant \|f\|_\infty. \tag{20}$$

If at least one between ω and ν is zero, then $\alpha = 0$ is the only choice that fulfils (18) and the statement is trivial. Now we suppose that both ω and ν are different from zero. It follows that the second derivative of the function $t \to (1-\omega\, t)^{-2\nu}$ is strictly positive in $[0,+\infty) \setminus \{\omega^{-1}\}$ while its first derivative at the origin is $2\,\omega\,\nu$ (see Figure 1 (a)). By thus we need $\alpha \leqslant 2\,\omega\,\nu$ and hence ω has to be positive. Therefore if $\|f\|_\infty \leqslant \omega^{-1}$ holds then $\alpha \leqslant 2\,\omega\,\nu$ ensures (20) and also (18) and this is point *1* of the Proposition. If $\omega^{-1} < \|f\|_\infty$ holds, then we also have to provide $1+\alpha\,\|f\|_\infty \leqslant 1/(1-\omega\,\|f\|_\infty)^{2\nu}$ in order to ensure (20), and this is point *2* of the Proposition.

Now we deal with the second part of the proposition. As we already did with (18), we write the scalar inequality (19) as a matrix inequality: $(I_{N(\boldsymbol{n})} - \omega A)^{\vartheta} A\,(I_{N(\boldsymbol{n})} - \omega A)^{\vartheta} \leqslant A - \beta A^2$, which is implied by the function inequality $(1-\omega f)^{\vartheta} f\,(1-\omega f)^{\vartheta} \leqslant f - \beta f^2$. As before, the assumption $0 \leqslant f$ allows one to write it as $(1-\omega f)^{2\vartheta} \leqslant 1-\beta f$ and we infer that (19) is implied by

$$(1-\omega t)^{2\vartheta} \leqslant 1-\beta t, \qquad 0 < t \leqslant \|f\|_\infty. \tag{21}$$

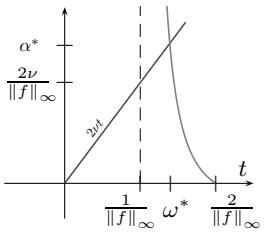
If at least one of ω and ϑ is zero, then the choice $\beta = 0$ fulfils (21) and then (19). Now we suppose that both ω and ν are different from zero. It follows that the second derivative of the function $t \to (1-\omega\, t)^{2\vartheta}$ is nonnegative for every $t \in \mathbb{R}$ (see Figure 1 (b)). By thus, if we want (21) to hold for

each value of t in $(0, \|f\|_\infty]$, we only have to require $1 - \beta \|f\|_\infty \geqslant \left(1 - \omega \|f\|_\infty\right)^{2\vartheta}$, and ω has to belong to $[0, 2/\|f\|_\infty]$ for $\beta \geqslant 0$. This completes the second part of the proof. □

We remark that Proposition 3 suggests the best[2] values for Richardson's parameters $\omega^{(\text{pre})}$ and $\omega^{(\text{post})}$ in the smoothing steps, i.e. the values ω that lead to the largest α and β respectively.

Presmoother If no iteration is performed ($\nu = 0$) we do not care of $\omega^{(\text{pre})}$, but we get $\alpha = 0$; otherwise we observe that the function

$$\left(\frac{1}{\|f\|_\infty}, \frac{2}{\|f\|_\infty}\right] \ni \omega \to \frac{\dfrac{1}{\left(1 - \omega \|f\|_\infty\right)^{2\nu}} - 1}{\|f\|_\infty}$$



decreases from $+\infty$ to zero, so there exists just one optimal parameter ω^* and it is such that $1/\|f\|_\infty < \omega^* < 2/\|f\|_\infty$ and $2\nu\omega^* = ((1 - \omega^*\|f\|_\infty)^{-2\nu} - 1)/\|f\|_\infty$. If we set $t = \omega\|f\|_\infty$ we get the polynomial equality $(2\nu t + 1)(1 - t)^{2\nu} = 1$, and its root t^* in the open interval $(1, 2)$ can be easily computed, or tabulated:

ν	1	2	3	4	5	6
t^*	1.5	1.606	1.670	1.715	1.747	1.772
$2\nu t^*$	3	6.423	10.02	13.72	17.47	21.27

so we get $\omega^{(\text{pre})} = t^*/\|f\|_\infty$ and $\alpha = 2\nu t^*/\|f\|_\infty$.

Postsmoother If no iteration is performed ($\nu = 0$) we do not care of $\omega^{(\text{post})}$ and get $\beta = 0$; otherwise $\omega_i^{(\text{post})} = 1/\|f_i\|_\infty$ gives the largest value for β, i.e. $\beta = 1/\|f\|_\infty$.

5.2 Approximation property

We first define the notation that we use in this section. Let $x \in \mathbb{R}^d$, the set of all "corners" is given by

$$\Omega(x) = \left\{ y_j \,|\, (y_j)_i \in \{x_i,\, \pi + x_i\},\, i = 1, \ldots, d,\, j = 1, \ldots, 2^d \right\}$$

[2] according to Ruge-Stüben theory [18].

which has cardinality 2^d. The set of "mirror points" (see [14]) is denoted as $\mathcal{M}(x) = \Omega(x) \setminus \{x\}$, e.g. for $d = 1$ it is $\mathcal{M}(x) = \{\pi + x\}$. Furthermore, we define

$$g[x] = (g(y_1), \ldots, g(y_{2^d})), \qquad y_j \in \Omega(x), \qquad j = 1, \ldots, 2^d \tag{22}$$

and its Euclidean norm is $\|g[x]\|_2^2 = \sum_{j=1}^{2^d} g(y_j)^2 = \sum_{y \in \Omega(x)} g(y)^2$.

We denote by S the fundamental set, which is $[0, \pi]^d$ for τ algebra and $[0, 2\pi]^d$ for circulant and Hartley algebra. To ensure the validity of the key assumptions (15c) we define p_i, the generating function of the projector, according to the following conditions. Let x_0 be the unique zero of f_i in S, $\forall x \in S$ we choose p_i such that

$$\limsup_{x \to x_0} \left| \frac{p_i(y)}{f_i(x)} \right| < +\infty, \qquad y \in \mathcal{M}(x),\ i = 0, \ldots, m-1, \tag{23}$$

where

$$0 < \sum_{y \in \Omega(x)} p_i^2(y), \qquad i = 0, \ldots, m-1. \tag{24}$$

From (24) we emphasize that the projector R_i is full rank. In the following proposition we prove that with the conditions (23) and (24) the assumption (15c) is verified and therefore the AMG defined in Section 4.1 is convergent.

Remark 3 The conditions (23) and (24) are a multidimensional generalization of the condition (4.2) in [1]. Condition (23) is stronger than the conditions (17) in [20] which come from the TGM optimality (relation (23) could require an higher degree of p_i).

Proposition 4 *Let $A = \mathcal{A}_{\boldsymbol{n}}(f)$ with $\mathcal{A} \in \{\mathcal{C}, \mathcal{H}, \tau\}$ and f be a d-variate nonnegative trigonometric polynomial with a single zero in the fundamental set. Let $R = K_{\boldsymbol{n}} \cdot \mathcal{A}_{\boldsymbol{n}}(p)$ as in Table* (3) *and define $CGC = I_{N(\boldsymbol{n})} - R^H (RAR^H)^{-1} RA$ as in* (5)*. If $p(x)$ fulfils* (23) *and* (24) *then there exists a real and positive value γ such that*

$$\|CGC\, \mathbf{x}\|_A^2 \leqslant \gamma \|\mathbf{x}\|_{A^2}^2, \qquad \mathbf{x} \in \mathbb{C}^{N(\boldsymbol{n})}. \tag{25}$$

Proof Defining $\hat{R} = R \cdot A^{1/2}$, the equation (25) is implied by

$$I_{\boldsymbol{n}} - (\hat{R})^H \left[\hat{R} (\hat{R})^H \right]^{-1} \hat{R} \leqslant \gamma A, \tag{26}$$

where $\hat{R} = K_{\boldsymbol{n}} \mathcal{A}_{\boldsymbol{n}}(\hat{p})$ with $\hat{p} = p f^{1/2}$. Inequality (26) is the same relation that can be found in [20,22] for the TGM convergence in the multilevel

case, but now we have $\hat{R}$ instead of R. Therefore, we can proceed as in [20] (pag. 23-25). Using tensor product arguments, we can extend the decomposition of $K_{N_i}Q_i$ given in Table 2 to the multidimensional case. Thanks to this decomposition, for each algebra exist a suitable permutation by rows and columns of $Q_{\boldsymbol{n}}^H(I_{\boldsymbol{n}} - (\hat{R})^H[\hat{R}(\hat{R})^H]^{-1}\hat{R})Q_{\boldsymbol{n}}$, such that it is a $2^d \times 2^d$ block diagonal matrix and a generic diagonal block is given by

$$I_{2^d} - \frac{1}{\|\hat{p}[w_{\boldsymbol{s}}^{[\boldsymbol{n}]}]\|_2^2}\hat{p}[w_{\boldsymbol{s}}^{[\boldsymbol{n}]}](\hat{p}[w_{\boldsymbol{s}}^{[\boldsymbol{n}]}])^T,$$

for $\boldsymbol{s} \in \mathcal{I}_{\boldsymbol{n}/2}$. Therefore the (26) is implied by

$$Z(x) \leqslant \gamma I_{2^d}, \qquad x \in S,$$

where,

$$Z(x) = [\mathrm{diag}(f[x])]^{-1/2}(I_{2^d} - \frac{1}{\|\hat{p}[x]\|_2^2}\hat{p}[x](\hat{p}[x])^T)[\mathrm{diag}(f[x])]^{-1/2}.$$

Therefore the proposition is proved if we show that the Hermitian matrix valued function $Z(x)$ is uniformly bounded in spectral norm, the latter being implied by the L^∞ boundedness of the generic entry of $Z(x)$. For $r \neq s$ we have

$$Z_{r,s}(x) = -\frac{\hat{p}(y_r)\hat{p}(y_s)}{\sqrt{f(y_r)f(y_s)}} \cdot \frac{1}{\|\hat{p}[x]\|_2^2}, \tag{27}$$

while

$$Z_{s,s}(x) = \sum_{y \in \mathcal{M}(y_s)} \frac{\hat{p}^2(y)}{f(y_s)} \cdot \frac{1}{\|\hat{p}[x]\|_2^2}. \tag{28}$$

The expression in (27) is equal to

$$Z_{r,s}(x) = -\frac{p(y_r)p(y_s)}{\sum_{y \in \Omega(x)} p^2(y)f(y)}, \tag{29}$$

that belongs to L^∞ thanks to (23) and (24). In this case also a less restrictive condition (like the usual TGM condition defined in [20,22]) is sufficient. The expression in (28) is equal to

$$Z_{s,s}(x) = \frac{\sum_{y \in \Omega(y_s)} p^2(y)f(y) - p^2(y_s)f(y_s)}{f(y_s)\sum_{y \in \Omega(x)} p^2(y)f(y)} \tag{30}$$

$$= \frac{1}{f(y_s)} - \frac{p^2(y_s)}{\sum_{y \in \Omega(x)} p^2(y)f(y)}. \tag{31}$$

If $s = 1$ then $y_s = x$, while if $s > 1$ then $y_s \in \mathcal{M}(x)$, moreover $\Omega(y_s) = \Omega(x)$. Therefore, if $s > 1$ from (23) and (24) we deduce $f(y_s) \neq 0$ that implies $f(y_s)^{-1} \in L^\infty$. The second addend in (31) belongs to L^∞ like the (29). If $s = 1$ the quantity in (30) is equal to

$$Z_{1,1}(x) = \frac{\sum_{y\in\mathcal{M}(x)} p^2(y)f(y)}{f(x)\sum_{y\in\Omega(x)} p^2(y)f(y)} = \frac{\sum_{y\in\mathcal{M}(x)} p^2(y)f(y)}{f^2(x)} \cdot \frac{1}{\sum_{y\in\Omega(x)} p^2(y)\frac{f(y)}{f(x)}}. \tag{32}$$

From (23) the first factor in (32) belongs to L^∞ (now the TGM condition is no longer sufficient) and the second factor in (32) belongs to L^∞ thanks to (24). □

5.3 Proof of optimality

In order to prove (17), in Propositions 3 and 4 we showed that values α_i, β_i and γ_i exist in $(0, +\infty)$ such that they ensure (15) and then (13) (i.e., the AMG (3) is convergent); such values depend on the function f_i (γ_i depends on p_i too) but not on the dimensions $\boldsymbol{n}_i$ neither on the number of grids m used in algorithm (3). Therefore, the (17) is ensured if $\{f_i\}$ converges uniformly to a function f_* and the constants α_*, β_* and γ_* related to f_* are positive. In this paragraph we will use the symbol $\xrightarrow{u}$ to represent uniform function convergence (with respect to the usual $\sup$ norm), e.g. $f_i \xrightarrow{u} f_*$.

In the following we will consider generating functions as

$$f(\mathbf{x}) = \mu \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \big[1 - \cos(x_r)\big]^q \cdot \psi^{(r)}(\mathbf{x}) \tag{33}$$

being $q \in \mathbb{N} \setminus \{0\}$, $\mu \geqslant 0$, $\psi^{(r)} \in \mathcal{F}_d$ and f positive in $[-\pi, \pi)^d \setminus \{\mathbf{0}\}$ and vanishing with order $2q$ around $\mathbf{0}$, i.e. $\psi^{(r)}(\mathbf{0}) > 0$, $r = 1, \ldots, d$. We take a particular choice for polynomials p_i:

$$p_i(\mathbf{x}) = \zeta_i \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + c \cdot \prod_{r=1}^{d} \big[1 + \cos(x_r)\big]^q \qquad (\zeta \in \mathbb{R}). \tag{34}$$

We emphasize that this choice (34) implies that all the functions f_i share the structure (33).

Lemma 2 *Assume that f_0 takes the form* (33)*:*

$$f_0(\mathbf{x}) = \mu_0 \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot \psi_0^{(r)}(\mathbf{x}), \qquad (35)$$

with $\mu_0 \in \mathbb{R}$ and $\psi_0^{(r)} \in \mathcal{F}_d$ for $r = 1,\dots,d$, and let p_i and $\{f_i\}_{i\in\mathbb{N}}$ be defined as in (34) *and* (8) *respectively ($f_{i+1} = \Psi_d(p_i^2 f_i)$).*
Then it holds that also f_i takes the form (33) *for all i, in detail*

$$f_i(\mathbf{x}) = \mu_i \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot \psi_i^{(r)}(\mathbf{x}), \qquad i \in \mathbb{N} \quad (36)$$

with $\{\mu_i\}_{i\in\mathbb{N}}$ and $\{\psi_i^{(r)}\}_{i\in\mathbb{N}}$, $r = 1,\dots,d$, defined as

$$\begin{cases} \mu_{i+1} \quad = 2^{-d}\big(\zeta_i + 2^{qd}c\big)^2 \mu_i \\ \psi_{i+1}^{(r)}(\mathbf{x}) = c^2\, \Psi_d\left[(\phi_{\boldsymbol{q}^{[r]}}\psi_i^{(r)})(\mathbf{x})\right] \overset{def}{=} c^2\, \Phi_{\boldsymbol{q}^{[r]}}\left[\psi_i^{(r)}(\mathbf{x})\right] \end{cases} \qquad i \in \mathbb{N}, \qquad (37)$$

where

$$\phi_{\boldsymbol{q}^{[r]}}(\mathbf{x}) = \left[\frac{1+\cos(x_r)}{2}\right]^q \prod_{\substack{j=1,\dots,d \\ j\neq r}} \big[1+\cos(x_j)\big]^{2q} \qquad (38)$$

and $\boldsymbol{q}^{[r]} = 2q\boldsymbol{e} - q\boldsymbol{e}_r$ is the degree of $\phi_{\boldsymbol{q}^{[r]}}$, where $\boldsymbol{e}_r$ is the r-th vector of the canonical basis of $\mathbb{R}^d$ and $\boldsymbol{e} = (1,\dots,1) \in \mathbb{N}^d$.

Remark 4 Choice (34) for p_i is fundamental to get uniform convergence of $\{f_i\}$ since it shows that the structure (33) is kept at each level, it is then enough to show $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$. Moreover, choice (34) satisfies (23) and (24) (refer to the following Lemma 3) and therefore the Proposition 4. Of course a different choice for p_i could still satisfy (23) and (24) (see e.g. [14, 22]) but no longer to preserve the structure (36) for $\{f_i\}$.

Lemma 3 *Let f be defined as in* (33) *and p_i as in* (34) *for $i = 0,\dots,m-1$. Then* (23) *and* (24) *hold true.*

Therefore, under the same assumptions of Lemma 3, by Proposition 4 it exists $\gamma_i > 0$ such that (15c) holds true.

From Lemma 2 and Remark 4 we obtain the main tools to show $f_i \xrightarrow{u} f_*$: it simply follows from $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$, but we still have to prove that the latter is true. The key is equation (37), which defines the d sequences $\{\psi_i^{(r)}\}_{i\in\mathbb{N}}$, $r = 1,\dots,d$.

The proof will act as follows: from Proposition 2 we have that $\{\partial\psi_i^{(r)}\}_i$ is bounded by $\boldsymbol{q}^{[r]}$ definitely, and by equations (37) and (9) we have that each step $\psi_i^{(r)} \to \psi_{i+1}^{(r)}$ is linear. Convergence $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$ can be shown in the finite dimension vector space $\mathbb{R}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$ by linear algebra tools (mainly resorting to the Perron-Frobenius theorem [27] applied to the matrix of the transformation having dominant eigenvalue equal to 1), and then $f_i \xrightarrow{u} f_*$ holds true with

$$f_*(\mathbf{x}) = \mu_* \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi_*^{(r)}(\mathbf{x})$$

whenever $\mu_i \to \mu_*$ holds in $\mathbb{R}$.

From a technical point of view, it is easier to work with $\mathbb{C}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$ than with $\mathbb{R}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$, being $\mathbb{C}_{\boldsymbol{\eta}}[\mathbf{x}]$ the vector space of d-variate trigonometric polynomials with complex coefficients and degree up to $\boldsymbol{\eta} \in \mathbb{N}^d$

$$\mathbb{C}_{\boldsymbol{\eta}}[\mathbf{x}] = \left\{ \sum_{|\boldsymbol{k}| \leqslant \boldsymbol{\eta}} a_{\boldsymbol{k}} \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle} \ \text{ s.t. } \ a_{\boldsymbol{k}} \in \mathbb{C} \right\}$$

since it is possible to use its canonical basis $\mathcal{B}_\eta[\mathbf{x}] = \bigcup_{|\boldsymbol{k}| \leqslant \boldsymbol{\eta}} \{e^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}\}$. Of course we need (see (37)) the Fourier coefficients of $\phi_{\boldsymbol{q}^{[r]}}(\mathbf{x})$ to get the entries of the matrix $c^2 M(\Phi_{\boldsymbol{q}^{[r]}})$ that represent the transform $\psi_i^{(r)} \to \psi_{i+1}^{(r)} = c^2 \Phi_{\boldsymbol{q}^{[r]}}(\psi_i^{(r)})$ with respect to $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$.

Monodimensional case ($d = 1$): it is already investigate in [1]. Indeed we already know (see [1]) that for $d = 1$ the Fourier coefficients $b_k^{(q)}$ of $\phi_{\boldsymbol{q}^{[1]}} \equiv \phi_q$ are given by

$$b_k^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1 + \cos(x)}{2}\right]^q \mathrm{e}^{-\mathbf{i}kx} \, \mathrm{d}x = \begin{cases} \dfrac{1}{4^q} \binom{2q}{q+k} & \text{if } |k| \leqslant q \\ 0 & \text{if } |k| > q \end{cases}. \quad (39)$$

The linear dependence of $\psi_{i+1}^{[1]}$ on $\psi_i^{[1]}$ is exploited by the following

Proposition 5 ([1]) *Assume $d = 1$ and let $M(\Phi_q)$ be the matrix related to the linear function $\Phi_{q^{[1]}} : \mathbb{C}_q[x] \longrightarrow \mathbb{C}_q[x]$ with respect to the basis $\mathcal{B}_q[x] = [\mathrm{e}^{-\mathbf{i}qx}, \ldots, \mathrm{e}^{\mathbf{i}qx}]$. The following three properties hold:*

1. $[M(\Phi_q)]_{k,j} = b^{(q)}_{2k-j} \geqslant 0$ *and*

$$M(\Phi_q) = \begin{bmatrix} b^{(q)}_{-q} & & & & & & \\ \vdots & \boxed{\begin{matrix} b^{(q)}_{1-q} & b^{(q)}_{-q} & & & \\ \vdots & & \ddots & \ddots & \\ b^{(q)}_{q} & b^{(q)}_{q-1} & & & b^{(q)}_{1-q} \\ & \ddots & \ddots & & \vdots \\ & & & b^{(q)}_{q} & b^{(q)}_{q-1} \end{matrix}} & b^{(q)}_{-q} \\ \vdots & & \vdots \\ & & b^{(q)}_{q} \end{bmatrix}_{-q:q\times -q:q}, \qquad (40)$$

thus the inner block of $M(\Phi_q)$ *fulfils the hypothesis of Perron-Frobenius* [27] *theorem (in the strong version);*

2. the dominant eigenvalue of $M(\Phi_q)$ *is* $1/2$ *and it is simple;*

3. there exists a dominant eigenvector $\mathbf{a}^{(q)} = [a^{(q)}_{-q}, \ldots, a^{(q)}_{q}]^T \in \mathbb{R}^{2q+1}$ *related to the eigenvalue* $1/2$ *such that*

(a) $\mathbf{a}^{(q)}_{-q} = \mathbf{a}^{(q)}_{q} = 0$*;*

(b) $\mathbf{a}^{(q)}_{j} > 0$ *if* $|j| < q$*;*

(c) $\sum_{|j|<q} \mathbf{a}^{(q)}_{j} = 1$.

Moreover the polynomial $\mathcal{B}_q[x] \cdot \mathbf{a}^{(q)} \in \mathbb{C}_q[x]$*, whose components (with respect to* $\mathcal{B}_q[x]$*) are* $\mathbf{a}^{(q)}$*, is real and positive in* $[-\pi, \pi)$.

Proposition 5 has been used in [1] ($d = 1$) with the choice $c^2 = 2$ as follows: the dominant eigenvalue of $c^2 M(\Phi_q)$ is 1 and it is simple; we can assume that exists $\bar{\imath} \in \mathbb{N}$ such that $\psi^{[1]}_{\bar{\imath}}(x)$ is in $\mathbb{C}_q[x]$ and has nontrivial component along $\mathcal{B}_q[x] \cdot \mathbf{a}^{(q)}$ (otherwise $\psi^{[1]}_i \xrightarrow{u} 0$) since[3] $\psi^{[1]}_i(0) = \psi^{[1]}_0(0)$ for all $i \in \mathbb{N}$. Then it follows $\psi^{[1]}_i \xrightarrow{u} \psi^{[1]}_0(0)\, \mathcal{B}_q[x] \cdot \mathbf{a}^{(q)}$.

Multidimensional case ($d > 1$): we extend the previous result in Proposition 5 to the case $d > 1$ to show $\psi^{[r]}_i \xrightarrow{u} \psi^{[r]}_*$. The first step is to compute the Fourier coefficients $b^{(\boldsymbol{q}^{[r]})}_{\boldsymbol{k}}$ of $\phi_{\boldsymbol{q}^{[r]}}$ for any d:

Lemma 4 *The Fourier coefficients* $b^{(\boldsymbol{q}^{[r]})}_{\boldsymbol{k}}$ *of* $\phi_{\boldsymbol{q}^{[r]}}$ *(given by* (38)*) are*

$$b^{(\boldsymbol{q}^{[r]})}_{\boldsymbol{k}} = 2^{2q(d-1)} \prod_{s=1}^{d} b^{(q^{[r]}_s)}_{k_s} \in \begin{cases} (0; +\infty) & \text{if } |\boldsymbol{k}| \leqslant \boldsymbol{q}^{[r]}, \\ \{0\} & \text{otherwise,} \end{cases} \qquad \boldsymbol{k} \in \mathbb{Z}^d$$

[3] $\psi^{[1]}_{i+1}(0) = \frac{c^2}{2^1}\left[\frac{1+\cos(0)}{2}\psi^{[1]}_i(0) + \frac{1+\cos(\pi)}{2}\psi^{[1]}_i(\pi)\right] = \psi^{[1]}_i(0)$.

thus

$$\phi_{\boldsymbol{q}^{[r]}}(\mathbf{x}) = \sum_{|\boldsymbol{k}|\leqslant \boldsymbol{q}^{[r]}} b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})} e^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}$$

holds true.

Lemma 4 allows us to generalize Proposition 5 to the multilevel case $d > 1$.

Proposition 6 *Assume $d \in \mathbb{N} \setminus \{0\}$, $r \in \{1, \dots, d\}$ and let $M(\Phi_{\boldsymbol{q}^{[r]}})$ be the matrix related to the linear function $\Phi_{\boldsymbol{q}^{[r]}} : \mathbb{C}_{\boldsymbol{q}^{[r]}}[\mathbf{x}] \longrightarrow \mathbb{C}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$ with respect to the basis $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}] = \bigotimes_{s=1}^{d} \mathcal{B}_{(\boldsymbol{q}^{[r]})_s}[x_s]$. The following three properties hold:*

1. $[M(\Phi_{\boldsymbol{q}^{[r]}})]_{\boldsymbol{i},\boldsymbol{j}} \geqslant 0$ *and*

$$M(\Phi_{\boldsymbol{q}^{[r]}}) = 2^{2q(d-1)} \cdot \bigotimes_{s=1}^{d} M\big(\Phi_{(\boldsymbol{q}^{[r]})_s}\big), \qquad r \in \{1, \dots, d\} \tag{41}$$

2. *the dominant eigenvalue of $M(\Phi_{\boldsymbol{q}^{[r]}})$ is $2^{2q(d-1)-d}$ and it is simple;*
3. *there exists a dominant eigenvector $\mathbf{a}^{(\boldsymbol{q}^{[r]})} \in \bigotimes_{s=1}^{d} \mathbb{R}^{2(\boldsymbol{q}^{[r]})_s+1}$ (to which we refer with the usual d-index notation, assuming the s-th index to range in $\{-(\boldsymbol{q}^{[r]})_s, \dots, (\boldsymbol{q}^{[r]})_s\}$) related to the dominant eigenvalue such that*
   (a) $\mathbf{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} = 0$ if $|\boldsymbol{j}_s| = (\boldsymbol{q}^{[r]})_s$ al least for an $s \in \{1, \dots, d\}$;
   (b) $\mathbf{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} > 0$ if $|\boldsymbol{j}| < \boldsymbol{q}^{[r]}$;
   (c) $\sum_{|\boldsymbol{j}|<\boldsymbol{q}^{[r]}} \mathbf{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} = 1$.

Moreover the polynomial $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}] \cdot \mathbf{a}^{(\boldsymbol{q}^{[r]})} \in \mathbb{C}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$, whose components with respect to $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$) are $\mathbf{a}^{(\boldsymbol{q}^{[r]})}$, is equal to $\prod_{s=1}^{d} \Big(\mathcal{B}_{(\boldsymbol{q}^{[r]})_s}[x_s] \cdot \mathbf{a}^{((\boldsymbol{q}^{[r]})_s)}\Big)$, and it is real and positive in $[-\pi, \pi)^d$.

Proof We refer to entries of $M\big(\Phi_{\boldsymbol{q}^{[r]}}\big)$ with usual d-index notation, with $\boldsymbol{k}$ row index and $\boldsymbol{j}$ column index, whose range is described by the inequalities $|\boldsymbol{k}|, |\boldsymbol{j}| \leqslant \boldsymbol{q}^{[r]}$. To get the entries of $M\big(\Phi_{\boldsymbol{q}^{[r]}}\big)$ with respect to the basis

$\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$, we compute the image of each vector in $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$:

$$\begin{aligned}\Phi_{\boldsymbol{q}^{[r]}}\left(\mathrm{e}^{\mathbf{i}\langle \boldsymbol{j}|\mathbf{x}\rangle}\right) &= \Psi_d\left(\phi_{\boldsymbol{q}^{[r]}}(\mathbf{x})\ \mathrm{e}^{\mathbf{i}\langle \boldsymbol{j}|\mathbf{x}\rangle}\right)\\ &= \Psi_d\left[\sum_{|\boldsymbol{k}|\leqslant \boldsymbol{q}^{[r]}} b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})}\ \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}+\boldsymbol{j}|\mathbf{x}\rangle}\right]\\ &= \Psi_d\left[\sum_{\boldsymbol{j}-\boldsymbol{q}^{[r]}\leqslant \boldsymbol{k}\leqslant \boldsymbol{j}+\boldsymbol{q}^{[r]}} b_{\boldsymbol{k}-\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})}\ \mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}\right]\\ &= \sum_{\left\lceil\frac{\boldsymbol{j}-\boldsymbol{q}^{[r]}}{2}\right\rceil\leqslant \boldsymbol{k}\leqslant\left\lfloor\frac{\boldsymbol{j}+\boldsymbol{q}^{[r]}}{2}\right\rfloor} b_{2\boldsymbol{k}-\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})}\mathrm{e}^{\mathbf{i}\langle \boldsymbol{k}|\mathbf{x}\rangle}\end{aligned}$$

and by thus the $(\boldsymbol{k},\boldsymbol{j})$-entry of $M\left(\Phi_{\boldsymbol{q}^{[r]}}\right)$ is

$$b_{2\boldsymbol{k}-\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} = 2^{2q(d-1)}\prod_{s=1}^{d} b_{2\boldsymbol{k}_s-\boldsymbol{j}_s}^{(\boldsymbol{q}_s^{[r]})} = \left[\frac{1}{2^{\frac{d}{2}+q(1-d)}}\right]^2\prod_{s=1}^{d} 2b_{2\boldsymbol{k}_s-\boldsymbol{j}_s}^{(\boldsymbol{q}_s^{[r]})}.$$

What we find is a block structure that is represented by a tensor product (see (40), (41)) between matrices as $M\left(\Phi_{(\boldsymbol{q}^{[r]})_s}\right)$, thus by Proposition 5 the largest eigenvalue of $M\left(\Phi_{\boldsymbol{q}^{[r]}}\right)$ is $2^{2q(d-1)-d}$ and it is simple. A related eigenvector is $\mathbf{a}^{(\boldsymbol{q}^{[r]})} = \bigotimes_{s=1}^{d}\mathbf{a}^{((\boldsymbol{q}^{[r]})_s)}$, and this fulfils requests *(a)*, *(b)* and *(c)* of point *3* by Proposition 5.
The result on $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]\cdot\mathbf{a}^{(\boldsymbol{q}^{[r]})}$ follows from properties of tensor product. □

Remark 5 In the following we fix the restrictor parameter in (34) as $c = 2^{\frac{d}{2}+q(1-d)}$. Therefore from Proposition 6 the maximum eigenvalue of the restriction of $c^2\Phi_{\boldsymbol{q}^{[r]}}$ to $\mathbb{C}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$ is 1, and it is simple. Furthermore, a related eigenvector is $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]\cdot\mathbf{a}^{(\boldsymbol{q}^{[r]})}$ and

$$\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)} = \psi^{(r)}(\mathbf{0})\,\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]\cdot\mathbf{a}^{(\boldsymbol{q}^{[r]})},$$

since $c^2\Phi_{\boldsymbol{q}^{[r]}}$ does not change the value at the origin ($c^2\Phi_{\boldsymbol{q}^{[r]}}(g(\mathbf{0})) = g(\mathbf{0})$ for each $g\in\mathcal{F}_d$) and $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]\cdot\mathbf{a}^{(\boldsymbol{q}^{[r]})}$ is the only eigenvector of $c^2\Phi_{\boldsymbol{q}^{[r]}}$ (apart rescaling) related to the eigenvalue 1 that is dominant and simple.

Finally, summarizing all the proposed results we obtain the following property of optimality for the algorithm $\mathcal{AMG}$ in (3).

Theorem 2 (optimality $\mathcal{AMG}$) *Let $t, m, q \in \mathbb{N} \setminus \{0\}$ with $t > m$ and assume $\boldsymbol{z} \in (\mathbb{N} \setminus \{0\})^d$ and $f \in \mathbb{R}_{\boldsymbol{z}}$ as in* (33)*:*

$$f(\mathbf{x}) = \mu \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \big[1 - \cos(x_r)\big]^q \cdot \psi^{(r)}(\mathbf{x})$$

such that f is positive valued in $[-\pi, \pi)^d \setminus \{\mathbf{0}\}$ and vanishes around $\mathbf{0}$ with order $2q$.

To solve the linear system $\mathcal{A}_{\boldsymbol{n}}(f)\,\mathbf{x} = \mathbf{b}$, $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}\}$ ($\mu = 0$ if $\mathcal{A} = \tau$) consider the algorithm $\mathcal{AMG}$ (3) *with the assumptions of Table 3 and with the following choices for $i = 0, \ldots, m-1$:*

1. *the projectors p_i are as in* (34)*:*

$$p_i(\mathbf{x}) = \zeta_i \cdot \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + c \cdot \prod_{r=1}^{d} \big[1 + \cos(x_r)\big]^q, \qquad c = 2^{\frac{d}{2}+q(1-d)},$$

   ($\zeta_i = 0$ if $\mathcal{A} = \tau$);
2. *the smoothers $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ are the Richardson method with $\nu_i + \vartheta_i = 1$ and the relaxation parameter chosen according to the Proposition 3.*

Then at least one in (17) *is satisfied and thanks to Theorem* (1) *there exists a constant* $\mathrm{const}(f) < 1$ *such that*

$$\|AMG_0\|_A \leqslant \mathrm{const}(f) < 1,$$

with AMG_0 defined in (6). In particular $\mathrm{const}(f)$ *depends only on f (i.e. on q, μ and $\{\psi_r\}_r$) but not on $\boldsymbol{n}$. Moreover each step of $\mathcal{AMG}$ has linear computational cost (i.e. $O(N(\boldsymbol{n}))$) and therefore the algorithm $\mathcal{AMG}$ is optimal in the sense of [2].*

Proof The proof is a consequence of the results stated through this paper.

The first step is to show convergence of the generating functions to a positive polynomial. From Lemma 2 it follows that all generating functions f_i share the structure of f as shown in (36), with μ_i and $\psi_i^{(r)}$ defined in (37) by means of c, $\{\zeta_i\}_{i\in\mathbb{N}}$ and of operator $\Phi_{\boldsymbol{q}^{[r]}}$. It follows that the degree $\boldsymbol{\partial}\psi_i^{(r)} \in \mathbb{N}^d$ of $\psi_i^{(r)}$ satisfies

$$\boldsymbol{\partial}\psi_{i+1}^{(r)} \leqslant \left\lfloor \frac{\boldsymbol{\partial}\psi_i^{(r)} + \boldsymbol{q}^{[r]}}{2} \right\rfloor, \qquad r \in \{1, \ldots, d\}, \quad i \in \mathbb{N},$$

thus by Proposition 2 $\boldsymbol{\partial}\psi_i^{(r)} \leqslant \boldsymbol{q}^{[r]}$ holds from a certain index $\bar{\imath}$ forward. We can then apply Proposition 6 and Remark 5 to get that each sequence $\{\psi_i^{(r)}\}_{i\in\mathbb{N}}$ has a limit $\psi_*^{(r)} \in \mathbb{R}_{\boldsymbol{q}^{[r]}}[\mathbf{x}]$, that is $\psi^{(r)}(\mathbf{0})\,\mathcal{B}_{\boldsymbol{q}^{[r]}}[\mathbf{x}] \cdot \mathbf{a}^{(\boldsymbol{q}^{[r]})}$.

We define $\zeta_i = 0$ if $\mathcal{A} = \tau$, while if $\mathcal{A} \in \{\mathcal{C}, \mathcal{H}\}$ we require $f_i(w_{\mathbf{0}}^{[\boldsymbol{n}_i]}) = f_i(w_{\boldsymbol{e}}^{[\boldsymbol{n}_i]})$ that means $\mu_i = f_i(w_{\boldsymbol{e}}^{[\boldsymbol{n}_i]})$, thus we define

$$\zeta_i = 2^{\frac{d}{2}} \left(\sqrt{\frac{\mu_{i+1}}{\mu_i}} - 2^q \right).$$

With this choice for ζ_i we have uniform convergence for f_i and $p_i(\mathbf{0}) = \zeta_i + c2^{qd} \neq 0$.

The second step is to check that all hypotheses of Theorem 1 are fulfilled (convergence) and that at least one in (17) holds (optimality). This means that we have to provide positive lower bounds for at least one between α_i/γ_i and β_i/γ_i, and these bounds have not to depend on the parameters t and m. It follows from Proposition 3 and uniform convergence of f_i that:

- if $\nu_i \geqslant 1$ a positive value α exists such that (15a) holds for each i,
- if $\vartheta_i \geqslant 1$ a positive values β exists such that (15b) holds for each i.

We consider now the (15c). With our choice for p_i, since $p_i(\mathbf{0}) \neq 0$, the Lemma 3 implies (23) and (24). Therefore from Proposition 4 and uniform convergence of f_i follow that a finite γ exists such that (15c) holds for each i. Finally, combining the previous results, at least one in (17) is fulfilled with $\alpha/\gamma > 0$ or $\beta/\gamma > 0$.

Finally, thanks to Lemma 1 each $\mathcal{AMG}$ iteration has a computational cost linear in N_0 and hence it is optimal in the sense of [2]. □

Remark 6 Theorem 2 ensures $\mathcal{AMG}$ optimality if just one smoother iteration is performed: if we choose $\nu_i + \vartheta_i$ larger on each grid i, we improve the convergence factor and optimality holds until $\nu_i + \vartheta_i \leqslant \text{const}(2^d - 1)^i$ according to Lemma 1.

Remark 7 The optimality results of Theorem 2 can be extended to the linear system $\mathcal{A}_{\boldsymbol{n}}(\tilde{f})\,\mathbf{x} = \mathbf{b}$ generated from a function $\tilde{f}$ with a zero shifted in $\bar{\mathbf{x}} \in \mathbb{R}^d$

$$\tilde{f}(\mathbf{x}) = f(\mathbf{x} - \bar{\mathbf{x}})$$

under a bit stronger assumptions: we require at least one of

1. $\mathcal{A} \in \{\mathcal{C}, \mathcal{H}\}$,
2. $\mathcal{A} = \tau$ and $\bar{\mathbf{x}} \in \{0, \pi\}^d$.

In both cases to keep $\tilde{f}_i(\mathbf{x}) = f_i(\mathbf{x} - 2^i\bar{\mathbf{x}})$ it suffices to define the polynomial projectors as

$$\tilde{p}_i(\mathbf{x}) = p_i(\mathbf{x} - 2^i\bar{\mathbf{x}})$$

and write

$$\begin{aligned}\tilde{f}_{i+1}(\mathbf{x}) &= \frac{1}{2^d}\sum_{\boldsymbol{s}\in\{0;1\}^d}(\tilde{p}_i^2\tilde{f}_i)\Big(\frac{\mathbf{x}}{2}+\pi\boldsymbol{s}\Big)\\ &= \frac{1}{2^d}\sum_{\boldsymbol{s}\in\{0;1\}^d}(p_i^2 f_i)\Big(\frac{\mathbf{x}-2^{i+1}\bar{\mathbf{x}}}{2}+\pi\boldsymbol{s}\Big)\\ &= f_{i+1}(\mathbf{x}-2^{i+1}\bar{\mathbf{x}}).\end{aligned}$$

6 Numerical experiments

For the application of the proposed AMG to restoration of blurred and noised images we refer the reader to [11,12]: for such a kind of problems we recall that the generating function usually vanishes or is very small in a neighborhood of (π,π). In this section we present some examples of two and three-level matrix algebra and Toeplitz linear systems. These experimentations confirm the optimality property proved in this paper for the multilevel algebra case and its possible extension to the multilevel Toeplitz matrices (the two grid optimality and the level independency have been already proved in [10,20]).

Two iterations of relaxed Richardson with $\omega^{(\text{pre})} = 1.6/\|f\|_\infty$ are used as pre-smoother and two iterations of relaxed Richardson with $\omega^{(\text{post})} = 1/\|f\|_\infty$ are used as post-smoother, according to Proposition 3. The initial solution is $\mathbf{x}^{(0)} = \mathbf{0}$ and the method is stopped when the relative norm of the residue is smaller than 10^{-7}. The algorithms are implemented in Fortran 90 using double precision.

6.1 Tau and circulant algebras

Here we consider coefficient matrices generated by the functions:

$$\begin{aligned}f(x,y) &= (4-2\cos(x)-2\cos(y))^2,\\ g(x,y) &= (4+2\cos(x)+2\cos(y))(8-\cos(x)-\cos(y)),\end{aligned}$$

f vanishes at the origin with order 4, while g vanishes at (π,π) with order 2. Therefore in the Circulant case we use the Strang correction and we replace f and g with their positive versions f_+ and g_+ defined as in (2). Furthermore, the conditioning number of $\mathcal{A}_{\boldsymbol{n}}(f)$ is $\mathcal{K}(\mathcal{A}_{\boldsymbol{n}}(f)) \simeq \mathcal{K}(\mathcal{A}_{\boldsymbol{n}}(g))^2 \simeq [\min_i(n_i)]^4$. We solve the linear system $\mathcal{A}_{\boldsymbol{n}}(z)\mathbf{x} = \mathbf{b}$ where $\boldsymbol{n} = (n_1,n_2)$, $z \in \{f,g\}$, $\mathcal{A} \in \{\tau,\mathcal{C}\}$ and the data vector $\mathbf{b}$ is obtained from the exact solution $\mathbf{x}$ taking four different types of solutions (constant, periodic, ...). As we can see from tables 4 and 5, according to the optimality proved

$N = n_1 \cdot n_2$	f s.t. $\mathbf{x}_i =$				g s.t. $\mathbf{x}_i =$			
	$\frac{i}{N}$	$(-1)^i$	$\cos\left(\frac{2i\pi}{N}\right)$	$(-\frac{i}{N})^i$	$\frac{i}{N}$	$(-1)^i$	$\cos\left(\frac{2i\pi}{N}\right)$	$(-\frac{i}{N})^i$
$(2^7-1)^2$	44	14	44	14	4	7	4	7
$(2^8-1)^2$	44	12	44	12	3	7	4	7
$(2^9-1)^2$	44	10	44	11	3	7	3	7
$(2^{10}-1)^2$	44	9	44	9	3	7	3	7

Table 4. Tau case: number of iterations increasing dimension $\boldsymbol{n} = (n_1, n_2)$ for $\tau_{\boldsymbol{n}}(f)$ and $\tau_{\boldsymbol{n}}(g)$, where $f(x,y) = (4 - 2\cos(x) - 2\cos(y))^2$ and $g(x,y) = (4 + 2\cos(x) + 2\cos(y))(8 - \cos(x) - \cos(y))$.

$N = n_1 \cdot n_2$	f_+ s.t. $\mathbf{x}_i =$				g_+ s.t. $\mathbf{x}_i =$			
	$\frac{i}{N}$	$(-1)^i$	$\cos\left(\frac{2i\pi}{N}\right)$	$(-\frac{i}{N})^i$	$\frac{i}{N}$	$(-1)^i$	$\cos\left(\frac{2i\pi}{N}\right)$	$(-\frac{i}{N})^i$
$(2^7)^2$	41	11	41	18	3	3	2	5
$(2^8)^2$	41	11	41	15	3	3	2	5
$(2^9)^2$	41	11	41	13	3	3	2	5
$(2^{10})^2$	41	11	41	11	3	3	2	5

Table 5. Circulant case: number of iterations increasing dimension $\boldsymbol{n} = (n_1, n_2)$ for $\mathcal{C}_{\boldsymbol{n}}(f_+)$ and $\mathcal{C}_{\boldsymbol{n}}(g_+)$, where f_+ and g_+ are $f(x,y) = (4 - 2\cos(x) - 2\cos(y))^2$ and $g(x,y) = (4 + 2\cos(x) + 2\cos(y))(8 - \cos(x) - \cos(y))$ plus them Strang corrections respectively.

in the previous sections, the proposed AMG converges with about a constant number of iterations when increasing the size of the problem. Furthermore, its asymptotically optimal value is low, it is already reached for small dimensions and it does not depend much on the decomposition in the frequencies space of the exact solution $\mathbf{x}$, stressing the robustness of our algorithm. Moreover, the solutions $\mathbf{x}_j = j/N(\boldsymbol{n})$ and $\mathbf{x}_j = \cos(2j\pi/N(\boldsymbol{n}))$, $j = 1, \ldots, N(\boldsymbol{n})$, have components essentially in the low frequencies and since in the case of $\mathcal{A}_{\boldsymbol{n}}(f)$ this is the ill-conditioned subspace, our AMG (like any general iterative method which can be used as smoother), converges more slowly; conversely, for $\mathcal{A}_{\boldsymbol{n}}(g)$ the low frequencies are the well-conditioned subspace and the method converges more quickly. For the solutions $\mathbf{x}_j = (-1)^j$ and $\mathbf{x}_j = (-j/N(\boldsymbol{n}))^j$, $j = 1, \ldots, N(\boldsymbol{n})$, we have the opposite behavior since they have larger components in the high frequencies.

We emphasize that our AMG maintains an optimal behavior varying the structure of the ill-conditioned subspace (it depends on the position of the zero) and the smoothness of the solution, while for the classic geometric multigrid the convergence depends strongly by these parameters (it requires a zero in the origin or a smooth solution). We remark that $\mathcal{A}_{\boldsymbol{n}}(f)$ is spectrally equivalent to the classic discretization of elliptic partial differ-

ential equations and usually these problems have smooth solutions. On the contrary, problems with coefficient matrix like $\mathcal{A}_{\boldsymbol{n}}(g)$ arise in some image restoration applications where the solution is the observed image that can have several jumps and therefore is only piecewise smooth. However our AMG is optimal for both smooth and high oscillating solutions.

6.2 The Toeplitz case

We consider now multilevel Toeplitz matrices. The technique that we use here to preserve the multilevel Toeplitz structure at each grid was proposed in [1]. It is based on the main idea of having a (multilevel) Toeplitz matrix at each recursion level by modifying the cutting matrices $K_{\boldsymbol{n}_i}$, $i = 1, \ldots, d$. Accordingly we define $K_{\boldsymbol{n}_i} = \bigotimes_{j=1}^{d} K_{(\boldsymbol{n}_i)_j}\{\boldsymbol{\kappa}_j\}$ where $\boldsymbol{\kappa} = \deg(p_0) - \boldsymbol{e}$ (we recall that the degree of p_i is constant for $i = 0, \ldots, m-1$) and

$$K_{(\boldsymbol{n_i})_j}\{\kappa\} = \left[0^{\kappa}_{(\boldsymbol{n}_{i+1})_j} \,\middle|\, K_{(\boldsymbol{n}_i)_j - 2\kappa} \,\middle|\, 0^{\kappa}_{(\boldsymbol{n}_{i+1})_j} \right] \in \mathbb{R}^{(\boldsymbol{n}_{i+1})_j \times (\boldsymbol{n}_i)_j}$$

where $0^{\kappa}_{(\boldsymbol{n}_{i+1})_j}$ is the null matrix of size $(\boldsymbol{n}_{i+1})_j \times \kappa$ and $K_{(\boldsymbol{n}_i)_j - 2\kappa}$ is the usual one-dimensional cutting matrix in the τ algebra defined in Table 2. For applying the AMG recursively, we must start with dimension $\boldsymbol{n}_0 = (2^t - 1)\boldsymbol{e} - 2\boldsymbol{\kappa}$ and the size on each grid is $\boldsymbol{n}_i = (2^{t-i} - 1)\boldsymbol{e} - 2\boldsymbol{\kappa}$.

In Table 6 we observe that the optimality holds in the Toeplitz case as well, for both f and g and for both smooth ($\mathbf{x}_j = j/N(\boldsymbol{n})$, for $j = 1, \ldots, N(\boldsymbol{n})$) and highly oscillating ($\mathbf{x}_j = (-1)^j$, for $j = 1, \ldots, N(\boldsymbol{n})$) solutions. We emphasize that such a kind of generating function f is not considered in the proof of level independency in [10] since f has a zero of order 4. The latter is an informal indication that it should be possible to extend this result of optimality also to the multilevel Toeplitz matrices with zeros of every finite order (see also [20]).

Finally, we compare our AMG for multilevel Toeplitz matrices with the other two techniques proposed in [16, 10] in order to extend an original idea by Fiorentino and Serra Capizzano in [14]. In §1 it was already observed that all these techniques are identically for nonnegative generating functions with a zero of order at most two: therefore they produce the same results on the function g and they are different only when applied to the function f. We note that the proposal in [16] requires to resort to the W-cycle and indeed the classical V-cycle does not converge in this case (function f, V-cycle and Richardson as smoother). On the other hand the method in [10] is only for generating function with zeros of order at most two, and consequently it is not necessary to consider low rank corrections in the computation of the matrices T_i. The only difference between our method and the one proposed in [10] relies on the choice of the projector: in this

$N = n_1 \cdot n_2$	g		f			
			our AMG		lin. interp.	
	$\frac{i}{N}$	$(-\frac{i}{N})^i$	$\frac{i}{N}$	$(-\frac{i}{N})^i$	$\frac{i}{N}$	$(-\frac{i}{N})^i$
$(2^7-\xi)^2$	3	6	119	25	54	14
$(2^8-\xi)^2$	3	6	120	20	72	12
$(2^9-\xi)^2$	3	6	121	17	87	11
$(2^{10}-\xi)^2$	3	6	121	13	115	9
$(2^{11}-\xi)^2$	3	6	121	10	134	7

Table 6. Toeplitz case: number of iterations for increasing dimension $\boldsymbol{n} = (n_1, n_2)$ for $T_{\boldsymbol{n}}(f)$ and $T_{\boldsymbol{n}}(g)$, where $f(x,y) = (4-2\cos(x)-2\cos(y))^2$, $g(x,y) = (4+2\cos(x)+2\cos(y))(8-\cos(x)-\cos(y))$. The two test solution have components $\mathbf{x}_i = \frac{i}{N}$ and $\mathbf{x}_i = (-\frac{i}{N})^i$ for $i = 1, \ldots, N$; $\xi = 3$ for our AMG with $f(x,y)$ while is one otherwise,

case (generating function f) the chosen projector is $R_i = K_{\boldsymbol{n}_i}\{0\}T_{\boldsymbol{n}_i}(p_i)$, with $p_i(x,y) = (1+\cos(x))(1+\cos(y))$ (the classic linear interpolation) for the method in [10] and $R_i = K_{\boldsymbol{n}_i}\{1\}T_{\boldsymbol{n}_i}(p_i^2)$ for our technique, $i = 0, \ldots, m-1$. In Table 6 we can see the optimal behavior of our method also for the generating function f, especially for the more significant case of a smooth solution ($\mathbf{x}_i = i/N(\boldsymbol{n})$, $i = 1, \ldots, N$). Conversely for the method in [10], in the case of a smooth solution, the number of iterations increases with the size of the problem since, as proved in Proposition 4, the choice of p_i is not enough to satisfy the approximation property when the function has a zero of order greater than two. The fact that the linear interpolation is not sufficient to obtain optimality is well-known also from the classic theory for the geometric multigrid, indeed $T_n(f)$ is spectrally equivalent to the discretization of the 4th derivative using finite differences of lower order and proper homogeneous boundary conditions (see [24]).

6.3 3D case

In this subsection we present for the first time a 3D example (a number of dimensions greater then two was previously considered only for the TGM case in [20] but without numerical experiments).

We consider functions that vanish in a generic point. The first test function is

$$h(x,y,z) = 3 - \cos\left(x - \frac{2}{3}\pi\right) - \cos\left(y - \frac{\pi}{3}\right) - \cos(z-1)$$

that vanishes in $(2\pi/3,\ \pi/3,\ 1)$ with order 2. The second test function is

$$r(x,y,z) = \big(1-\cos(x)\big)^2 + \big(1-\cos(y)\big)^2 + \big(1-\cos(z)\big)^2$$

$N = n_1 \cdot n_2 \cdot n_3$	Circulant				Toeplitz			
	h s.t. $\mathbf{x}_i =$		r s.t. $\mathbf{x}_i =$		h s.t. $\mathbf{x}_i =$		r s.t. $\mathbf{x}_i =$	
	$\frac{i}{N}$	$(-1)^i$	$\frac{i}{N}$	$(-1)^i$	$\frac{i}{N}$	$(-1)^i$	$\frac{i}{N}$	$(-1)^i$
$(2^4-\xi)^3$	8	4	31	7	9	7	82	13
$(2^5-\xi)^3$	8	4	31	7	8	7	92	9
$(2^6-\xi)^3$	7	4	31	7	7	6	96	6
$(2^7-\xi)^3$	7	4	31	7	7	6	99	4

Table 7. 3D case: number of iterations for increasing dimension $\boldsymbol{n} = (n_1, n_2, n_3)$, where $h(x,y,z) = 3-\cos(x-\frac{2}{3}\pi)-\cos(y-\frac{\pi}{3})-\cos(z-1)$, and $r(x,y,z) = (1-\cos(x))^2+(1-\cos(y))^2+(1-\cos(z))^2$. In the Circulant case $\xi = 0$, while in the Toeplitz case $\xi = 1$ for h and $\xi = 3$ for r.

that vanishes in $(0,0,0)$ with order 4.

The true solution has both smooth and highly oscillating components. In Table 7 we can note, according to the previous theory, the optimality behavior of the proposed AMG in the circulant case. Furthermore also in the Toeplitz case, using the technique described in the previous subsection, our AMG seems to maintain about a constant number of iterations increasing the size of the problem. This is a good news especially for the second test function r, since it has a zero of order 4 and, in order to preserve the Toeplitz structure at each level, in the restriction process we have to remove $2n_2n_3$ components in more with respect to the circulant case (i.e. it is neglected a substantial level of information).

7 Conclusion

In this paper we have proved the V-cycle optimality of the proposed AMG for coefficient matrices generated by a real and nonnegative multivariate polynomial f and belonging to multilevel matrix algebras like circulant, tau or Hartley. The AMG considered here is an extension of that proposed by Fiorentino and Serra in [14]: now the projector has to satisfy the more strictly conditions (23) and (24) (analogous conditions was proposed in [1] with a V-cycle analysis only in the onedimensional case). Concerning the future work, the main point to investigate is the extension of this proof to multidimensional Toeplitz matrices. Preliminary results in this direction can be found in [7, 10] for the level independency in the case of generating function with zeros of order at most 2, and in [20] for the TGM algorithm and implicitly for the level independency.

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Appendix

In this appendix we provide the proofs of the lemmas previously introduced.

Lemma 1

Proof By point *3* all matrices $A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i)$ involved in the $\mathcal{AMG}$ algorithm (3) are uniformly banded at each level and there exists a constant B_A (B stands for "band") such that each product $A_i\mathbf{x}$ (step 3.**2**) has computational cost less then $B_A \cdot N(\boldsymbol{n}_i)$. If we use relaxed Richardson with parameters $\omega_i^{(\text{pre})}$ and $\omega_i^{(\text{post})}$ then (4) becomes

$$\begin{cases} \mathcal{S}_i(\mathbf{x}) = \mathbf{x} + \omega_i^{(\text{pre})}(\mathbf{b}_i - A_i\mathbf{x}), \\ \widetilde{\mathcal{S}}_i(\mathbf{x}) = \mathbf{x} + \omega_i^{(\text{post})}(\mathbf{b}_i - A_i\mathbf{x}), \end{cases}$$

thus analogous considerations hold for smoothers (steps 3.**1** and 3.**7**) as well. Moreover projectors R_i are sparse too, because of $R_i = K_{\boldsymbol{n}_i}\,\mathcal{A}_{\boldsymbol{n}_i}(p_i)$ and $p_i \in \mathbb{R}_{\boldsymbol{q}}$, so there exists a constant B_R such that the cost of each product $R_i\,\mathbf{r}_i$ and ${R_i}^H\mathbf{x}_{i+1}^{(\text{out})}$ (steps 3.**3** and 3.**6**) will be less than $B_R \cdot N(\boldsymbol{n}_i)$. We emphasize that our choice on $\boldsymbol{q}$ will depend only on f_0, so B_A and B_R will depend on f_0. Because of fast size reduction on each level, smoothing iterations can be enriched (see also [22]) when i grows, at least up to $\nu_i + \vartheta_i \leqslant h \cdot (2^d - 1)^i$ (where $h \geqslant 1$ is a constant), and the cost $C_{\mathcal{AMG}}$ of each

iteration of algorithm $\mathcal{AMG}$ (3) still keeps linear:

$$
\begin{aligned}
C_{\mathcal{AMG}} &\leqslant \sum_{i=0}^{m-1} \left\{ \underbrace{(\nu_i+\vartheta_i)(B_A+3)}_{\text{steps } (\mathbf{1},\mathbf{7})} + \underbrace{B_A+1}_{\text{step } (\mathbf{2})} + \underbrace{2B_R+1}_{\text{steps } (\mathbf{3},\mathbf{6})} \right\} N(\boldsymbol{n}_i) + \underbrace{N(\boldsymbol{n}_m)^3}_{\text{case } i=m} \\
&< \sum_{i=0}^{+\infty} \left\{ h\,(2^d-1)^i\,(B_A+3) + 2 + B_A + 2B_R \right\} \frac{N(\boldsymbol{n}_0)}{2^{d\,i}} + N(\boldsymbol{n}_m)^3 \\
&= \left\{ h\,2^d(B_A+3) + \frac{2^d}{2^d-1}\,(2+B_A+2B_R) \right\} N(\boldsymbol{n}_0) + N(\boldsymbol{n}_m)^3 \\
&\leqslant 2^d \Big[2 + 3h + (1+h)B_A + 2B_R \Big] N(\boldsymbol{n}_0) + N(\boldsymbol{n}_m)^3
\end{aligned}
$$

and $N(\boldsymbol{n}_m)^3$ is small and it holds $[N(\boldsymbol{n}_m)]^3 \leqslant N(\boldsymbol{n}_0)$ if $m \geqslant 2t/3$, with t defined in Table 3.

Lemma 2

Proof We first suppose $\mu_0 = \zeta_i = 0$ and show (36) by induction. The initial step $i = 0$ is true for free. In order to prove the inductive step we assume that (36) holds for a given $i \in \mathbb{N}$. By replacing expressions of p_i (34) and of f_i (36) in (8), we find

$$
(p_i^2 f_i)(\mathbf{x}) = \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot \psi_i^{(r)}(\mathbf{x}) \cdot c^2 \prod_{j=1}^{d} \big[1+\cos(x_j)\big]^{2q}
$$

$$
= \sum_{r=1}^{d} 2^q \sin^{2q}(x_r)\, \psi_i^{(r)}(\mathbf{x}) \cdot \left(\frac{c^2}{2^q} \big[1+\cos(x_r)\big]^q \prod_{\substack{j=1,\dots,d \\ j\neq r}}{}_j \big[1+\cos(x_j)\big]^{2q} \right)
$$

$$
= \sum_{r=1}^{d} \big[1-\cos(2x_r)\big]^q \cdot c^2 \cdot \big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)(\mathbf{x}).
$$

Therefore we have

$$
(p_i^2 f_i)\Big(\frac{\mathbf{x}}{2} + \pi\boldsymbol{s}\Big) = \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot c^2 \cdot \big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)\Big(\frac{\mathbf{x}}{2} + \pi\boldsymbol{s}\Big)
$$

for all $\boldsymbol{s} \in \{0;1\}^d$. This equality allows us to find the following expression for f_{i+1}:

$$\begin{aligned} f_{i+1}(\mathbf{x}) &= \Psi_d\big[(p_i^2 f_i)(\mathbf{x})\big] \\ &= \frac{1}{2^d} \sum_{\boldsymbol{s}\in\{0;1\}^d} \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot c^2 \cdot \big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)\Big(\frac{\mathbf{x}}{2}+\pi\boldsymbol{s}\Big) \\ &= \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot \left[\frac{c^2}{2^d} \sum_{\boldsymbol{s}\in\{0;1\}^d} \big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)\Big(\frac{\mathbf{x}}{2}+\pi\boldsymbol{s}\Big)\right] \\ &= \sum_{r=1}^{d} \big[1-\cos(x_r)\big]^q \cdot c^2 \cdot \Psi_d\left[\big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)(\mathbf{x})\right] \end{aligned}$$

and this completes the inductive step.
Case $\mu_0,\, \zeta_i \in \mathbb{R}$ is straightforward because of

$$(p_i^2 f_i)(\mathbf{x}) = (p_i^2 f_i)(\mathbf{0})\cdot\chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) + \sum_{r=1}^{d} \big[1-\cos(2x_r)\big]^q \cdot c^2 \cdot \big(\phi_{\boldsymbol{q}^{[r]}}\, \psi_i^{(r)}\big)(\mathbf{x}),$$

$$\begin{aligned} \Psi_d[(p_i^2 f_i)(\mathbf{0})\cdot\chi_{2\pi\mathbb{Z}^d}(\mathbf{x})] &= \big(\big(\zeta_i + 2^{qd}c\big)\big)^2 \mu_i \cdot \Psi_d[\chi_{2\pi\mathbb{Z}^d}(\mathbf{x})] \\ &= 2^{-d}\big(\zeta_i + 2^{qd}c\big)^2 \mu_i \chi_{2\pi\mathbb{Z}^d}(\mathbf{x}) \end{aligned}$$

and thanks to the linearity of Ψ_d. □

Lemma 3

Proof The (24) is easy to prove since $\sum_{\mathbf{y}\in\Omega(\mathbf{0})} p_i^2(\mathbf{y}) \geqslant p_i^2(\mathbf{0})$ and $p_i(\mathbf{0}) = \zeta_i + c2^{qd}$. Therefore it is enough that $c \neq -\zeta_i 2^{-qd}$.

Concerning (23), let $\boldsymbol{\psi_i} = [\psi_i^{(1)}, \ldots, \psi_i^{(d)}]$, we show that

$$\limsup_{\mathbf{x}\to\mathbf{0}} \left|\frac{p_i(\mathbf{x}+\pi\boldsymbol{s})}{f_i(\mathbf{x})}\right| = \frac{c\, 2^{q(d-1)}}{\langle \boldsymbol{s} \,|\, \boldsymbol{\psi_i}(\mathbf{0})\rangle} < +\infty, \qquad \forall \boldsymbol{s} \in \{0,1\}^d \setminus \mathbf{0},$$

whenever $c > 0$ and $\psi_i^{(r)}(\mathbf{0}) > 0$ for $r = 1, \ldots, d$. This is implied by the following limit on the reciprocal:

$$\liminf_{\substack{\mathbf{x}\to\mathbf{0} \\ p_i(\mathbf{x}+\pi\boldsymbol{s})\neq 0}} \left|\frac{f_i(\mathbf{x})}{p_i(\mathbf{x}+\pi\boldsymbol{s})}\right| = \begin{cases} +\infty & \text{if } \sum_{r=1}^d s_r > 1, \\ \dfrac{\langle \boldsymbol{s} \,|\, \boldsymbol{\psi_i}(\mathbf{0})\rangle}{c\, 2^{q(d-1)}} & \text{if } \sum_{r=1}^d s_r \leqslant 1 \end{cases}. \qquad (42)$$

Therefore to satisfy the (23) it is enough to prove the (42). It holds

$$\liminf_{\substack{\mathbf{x}\to\mathbf{0}\\ p_i(\mathbf{x}+\pi\boldsymbol{s})\neq 0}} \left|\frac{f_i(\mathbf{x})}{p_i(\mathbf{x}+\pi\boldsymbol{s})}\right| = \tag{43}$$

$$= \frac{1}{c}\liminf_{\substack{\mathbf{x}\to\mathbf{0}\\ p_i(\mathbf{x}+\pi\boldsymbol{s})\neq 0}} \sum_{r=1}^{d} \frac{\left[1-\cos(x_r)\right]^q}{[1+\cos(x_r+\pi s_r)]^q}\cdot\frac{\psi_i^{(r)}(\mathbf{x})}{\prod\limits_{\substack{j=1,\dots,d\\ j\neq r}}[1+\cos(x_j+\pi s_j)]^q}$$

$$\geqslant \frac{1}{c}\sum_{r=1}^{d} \liminf_{\substack{\mathbf{x}\to\mathbf{0}\\ p_i(\mathbf{x}+\pi\boldsymbol{s})\neq 0}} \frac{\left[1-\cos(x_r)\right]^q}{[1+\cos(x_r+\pi s_r)]^q}\cdot\frac{\psi_i^{(r)}(\mathbf{x})}{\prod\limits_{\substack{j=1,\dots,d\\ j\neq r}}[1+\cos(x_j+\pi s_j)]^q}.$$

Each term of the last sum is nonnegative around $\mathbf{0}$, and at least two are infinite whenever two or more components of $\boldsymbol{s}$ are equal to 1.
If just one component of $\boldsymbol{s}$ is equal to 1, that is $\boldsymbol{s} = \boldsymbol{e}_{\overline{r}}$, then from the second equality of (43) we get

$$\liminf_{\substack{\mathbf{x}\to\mathbf{0}\\ p_i(\mathbf{x}+\pi\boldsymbol{e}_{\overline{r}})\neq 0}} \left|\frac{f_i(\mathbf{x})}{p_i(\mathbf{x}+\pi\boldsymbol{e}_{\overline{r}})}\right| =$$

$$= \frac{\psi_i^{(\overline{r})}(\mathbf{0}) + \liminf\limits_{\substack{\mathbf{x}\to\mathbf{0}\\ x_{\overline{r}}\neq 0}} \sum\limits_{\substack{r=1,\dots,d\\ r\neq\overline{r}}} \left[\frac{1-\cos(x_r)}{1-\cos(x_{\overline{r}})}\right]^q \psi_i^{(r)}(\mathbf{x})}{c\,2^{q(d-1)}} = \frac{\psi_i^{(\overline{r})}(\mathbf{0})}{c\,2^{q(d-1)}}.$$

The proof is trivial if $\boldsymbol{s} = \mathbf{0}$ since $p(\mathbf{x}+\pi\boldsymbol{s})$ converges to $2^{qd} > 0$ if $\mathbf{x}$ goes to $\mathbf{0}$, while $f(x)$ vanishes. □

Lemma 4

Proof The proof follows from Fubini's theorem for rectangles, with the advantage that $\phi_{\boldsymbol{q}^{[r]}}$ can be factorized thanks to (38). For any $\boldsymbol{k} \in \mathbb{Z}^d$ it

holds

$$\begin{aligned} b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})} &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d}\phi_{\boldsymbol{q}^{[r]}}(\mathbf{x})e^{-\mathbf{i}\langle\boldsymbol{k}|\mathbf{x}\rangle}\mathrm{d}\mathbf{x} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1+\cos(x_r)}{2}\right]^q e^{-\mathbf{i}k_r x_r}\mathrm{d}x_r \quad \cdot \\ &\quad \cdot \prod_{\substack{s=1,\dots,d\\ s\neq r}}\frac{1}{2\pi}\int_{-\pi}^{\pi}[1+\cos(x_s)]^{2q}e^{-\mathbf{i}k_s x_s}\mathrm{d}x_s \\ &= b_{k_r}^{(q)}\cdot\prod_{\substack{s=1,\dots,d\\ s\neq r}}2^{2q}b_{k_s}^{(2q)} = 2^{2q(d-1)}\prod_{s=1}^{d}b_{k_s}^{(q_s^{[r]})} \end{aligned}$$

which completes the proof. □