Multigrid Methods for Strongly Ill-Conditioned Structured Matrices

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Abstract

Multigrid methods are highly efficient solution techniques for large sparse linear systems which are positive definite and ill-conditioned. Matrices belonging to the two-level Toeplitz class or to one of the two-level trigonometric matrix algebras are associated with generating functions. In this paper, we develop multigrid methods for linear systems which correspond to generating functions with whole zero curves. For functions with isolated zeros several multigrid methods are well-known, but the case of zero curves is significantly more difficult. First, we develop a two-grid method based on the Galerkin approach, extending results from the isolated zero case. We prove two-grid convergence, and describe why a Galerkin-based multigrid method is not practical. Instead, we use a rediscretization technique which is based on a sufficiently accurate approximation of the zero curve on the coarse grid. With this approach the same convergence behavior as for the Galerkin method is obtained, but matrices remain sparse on coarser levels, allowing several levels to be used.

Key words: multilevel Toeplitz, trigonometric matrix algebras, multigrid methods, iterative methods
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1 Introduction

Multigrid methods belong to the fastest iterative methods for the solution of large sparse and structured linear systems of equations. They can either be used as stand-alone solvers or as preconditioners for Krylov subspace methods. Many applications such as discretization of PDEs lead to two-level Toeplitz systems. If these are associated with strictly positive generating functions, they are easily solved with the conjugate gradient method. If the generating function has up to a finite number of zeros, the use of multigrid methods is significantly more efficient [7,5] . Multigrid methods have also been developed for sparse matrices belonging to a trigonometric algebra such as circulant or tau matrices [2,4] . However, discretization of many PDEs such as the Helmholtz equation leads to two-level Toeplitz systems whose underlying generating function has a whole zero curve. If these

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systems are solved with normal equations, the classical convergence theory for (multilevel) structured matrices [5,3,2] does not hold anymore, and standard multigrid methods fail. Starting from a Galerkin method, we present a multigrid algorithm which is based on rediscretization on coarser levels and on approximation of the zero curve. Since such a method preserves the bandedness of a given matrix, it is more suitable for practical use than a pure Galerkin method.

The article is organized as follows. In Chapter 2 we explain the correspondence between certain classes of structured matrices and generating functions. Moreover, we give an overview of well-known multigrid methods for structured linear systems whose generating functions have isolated zeros. In Chapter 3 we present a Galerkin-based multigrid method for linear systems corresponding to functions with whole zero curves. After giving a convergence proof for the two-grid method (TGM), we explain the practical problems of extending the TGM to a multigrid method (MGM). The MGM devised in Chapter 4 uses the same restriction and prolongation matrices as the Galerkin method, but coarse grid matrices are obtained by approximating the zero curve on coarser levels. This approach results in similar iteration numbers as the Galerkin method, but the computational cost of one iteration is optimal, i.e. of the same order as one matrix-vector product. Numerical results will be given to illustrate and compare the different multilevel methods.

2 Multigrid methods and generating functions

Generating functions are closely related to two-level Toeplitz (or BTTB) matrices [8,10]. Suppose $f(x,y)$ is a real-valued Lebesgue integrable function which is defined on $[-\pi,\pi]^2$, and periodically extended on the whole plane. Then the Fourier coefficients of f are given by

$$t_{k,l} = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)e^{-ikx-ily}dx\,dy \quad (k,l\in\mathbb{Z}).$$

$A_{mn}[f]$ is the corresponding mn-by-mn BTTB matrix with entries $(A_{mn}[f])_{(j,k)(p,q)} = t_{j-k,p-q}$ $(0\le j,k<m\,,\ 0\le p,q<n)$, where (j,k) indicates the block in $A_{mn}[f]$ and (p,q) the position within the block. If f_{min} and f_{max} denote the infimum and supremum values of f, and if $f_{min}<f_{max}$, then for all $m,n\ge 1$, the eigenvalues of $A_{mn}[f]$ lie in the interval (f_{min},f_{max}). For $n,m\to\infty$, the extreme eigenvalues tend to f_{min} and f_{max}.
Matrices belonging to a two-level trigonometric algebra are diagonalized by a unitary transform Q_{mn}, i.e. they are of the form

$$A_{mn}[f] = Q_{mn}^H\cdot\Lambda_{mn}[f]\cdot Q_{mn}\quad, \tag{1}$$

where $\Lambda_{mn}[f]$ is the diagonal matrix containing the eigenvalues $\lambda_{k,l}$ of $A_{mn}[f]$. For two-level circulant matrices these are given by $\lambda_{k,l}=f(\frac{2\pi l}{m},\frac{2\pi k}{n})$ with $0\le k\le n-1, 0\le l\le m-1$, and for two-level tau matrices by $\lambda_{k,l}=f(\frac{\pi l}{m+1},\frac{\pi k}{n+1})$ with $1\le k\le n, 1\le l\le m$. This implies that unlike Toeplitz matrices, matrices from trigonometric algebras can become singular if f is zero at one of the grid points. If this happens, $A_{mn}[f]$ is usually replaced by the so called Strang correction [11], which was originally defined for circulant matrices, but which can be used for other matrix algebras as well [2].

For Toeplitz, tau, and circulant matrices corresponding to functions with isolated zeros multigrid methods turned out to be the fastest iterative solvers. All methods developed so far are based on the algebraic multigrid method of Ruge and Stüben [9]. Usually, a simple smoother S such as the damped Richardson or Jacobi method is applied. After defining the restriction matrix R, the coarse-grid matrix A_C is computed with the Galerkin approach $A_C = RA_{mn}[f]R^H$, and thus the coarse grid correction X with $X = I - R^H A_C^{-1} RA_{mn}[f]$. If ν_1 denotes the number of presmoothing steps and ν_2 the number of postsmoothing steps, the iteration matrix TG of the TGM is $TG = S^{\nu_2} X S^{\nu_1}$. The TGM is extended to an MGM by recursively using the TGM scheme for A_C instead of inverting A_C exactly. Ruge and Stüben have proved convergence of the TGM [9]. In order to state their theorem we must define, for an arbitrary matrix A, the following inner products in addition to the Euclidean inner product $\langle u, v\rangle$:

$$\langle u,v\rangle_0 = \langle diag(A)u, v\rangle \, , \; \langle u,v\rangle_1 = \langle Au, v\rangle \, , \; \langle u,v\rangle_2 = \langle diag(A)^{-1}Au, Av\rangle \tag{2}$$

The respective norms, which are derived from these inner products, are denoted $\|\cdot\|_i$, $i = 0, 1, 2$. Moreover, let $\nu_1 = 0$ and $\nu_2 = 1$.

Theorem 1 (Ruge, Stüben,[9])
Let A be a positive definite mn-by-mn matrix, and let S be a smoother satisfying the smoothing condition, i.e. there exists an $\alpha > 0$ such that

$$\|Sx\|_1^2 \leq \|x\|_1^2 - \alpha\|x\|_2^2 \, , \quad \forall x \in \mathbb{R}^{mn} \, . \tag{3}$$

Furthermore, suppose that the restriction operator R has full rank and that the correcting condition is satisfied, i.e. there exists a scalar $\beta > 0$ such that

$$\min_{y \in \mathbb{R}^{m_C n_C}} \|x - R^H y\|_0^2 \leq \beta \|x\|_1^2 \, , \quad \forall x \in \mathbb{R}^{mn} \, . \tag{4}$$

Then $\beta > \alpha$, and the convergence factor of the two-level method $\|TG\|_1$ is bounded by $\|TG\|_1 \leq \sqrt{1 - \alpha/\beta}$

In the following, we wish to describe TGM and MGM in terms of generating functions. The restriction is split into two parts $R = B \cdot E$, where B is in the same class as $A_{mn}[f]$, dealing with the zeros of f, and E is the elementary restriction matrix of the class. Then computation of the coarse grid matrix A_C is done with

$$A_C = E \cdot (B \cdot A_{mn}[f] \cdot B^H) \cdot E^H \, . \tag{5}$$

Since for circulant or tau matrices $\hat{A} = BA_{mn}[f]B^H$ is still in the circulant or tau class, this inner multiplication in (5) is directly translated to generating functions:

$$\hat{f}(x, y) = f(x, y) \cdot b^2(x, y) \, . \tag{6}$$

Elementary restriction represents the spectral link between the space of frequencies on the fine and on the coarse grid. With the choice

$$E_n^{(\mathrm{circ})} = \begin{pmatrix} 1\;0 & & & \\ & 1\;0 & & \\ & & \ddots \;\; \ddots & \\ & & & 1\;0 \end{pmatrix} \, , \quad E_n^{\mathrm{tau}} = \begin{pmatrix} 0\;1\;0 & & & \\ & 0\;1\;\;0 & & \\ & & \ddots \;\; \ddots \;\; \ddots & \\ & & & 0\;\;1\;0 \end{pmatrix} \tag{7}$$

of the one-level restriction matrices, we obtain $E = E_m^{\mathrm{circ}} \otimes E_n^{\mathrm{circ}}$ or $E = E_m^{\mathrm{tau}} \otimes E_n^{\mathrm{tau}}$, respectively. m, n must be even in the circulant case, leading to $n_C = n/2, m_C = m/2$ on the coarse grid, and odd in the tau case, resulting in $n_C = \lfloor n/2 \rfloor, m_C = \lfloor m/2 \rfloor$. With this choice of E, computation of $A_C = E \cdot \hat{A} \cdot E^H$ is translated to

$$f_2(x,y) = \frac{1}{4}\left(\hat{f}(\frac{x}{2}, \frac{y}{2}) + \hat{f}(\frac{x}{2} + \pi, \frac{y}{2}) + \hat{f}(\frac{x}{2}, \frac{y}{2} + \pi) + \hat{f}(\frac{x}{2} + \pi, \frac{y}{2} + \pi)\right) \quad . \tag{8}$$

for circulant matrices. For tau matrices $x/2+\pi$ and $y/2+\pi$ are replaced by $\pi - x/2$ and $\pi - y/2$. This means f_2 is obtained from the Fourier series of $\hat{f}$ by picking every second coefficient in x and every second coefficient in y. For BTTB matrices the same correspondence between matrices and functions can be used. However, it is important to note that in general the matrices $\hat{T}$ and T_C are not BTTB, but a sum of a BTTB matrix and a matrix of rank $O(n+m)$. In [3,2] different cutting techniques are presented to make sure that the coarse grid matrix still has two-level Toeplitz structure. For some important special cases it can be shown that the coarse grid matrix obtained without additional cutting is still BTTB. For generating functions $f(x,y)$ with isolated zeros the choice of $b(x,y)$ is based on the following idea, which results in a coarse grid function $f_2(x,y)$ with similar properties as f. If f has a zero at $\mathbf{x_0} = (x_0, y_0)$, then b should be nonzero at $\mathbf{x_0}$ and zero at the so called mirror points, which are

$$M(\mathbf{x_0}) = \{(x_0+\pi, y_0), (x_0, y_0+\pi), (x_0+\pi, y_0+\pi)\} \tag{9}$$

in the circulant case and

$$M(\mathbf{x_0}) = \{(\pi - x_0, y_0), (x_0, \pi - y_0), (\pi - x_0, \pi - y_0)\} \tag{10}$$

in the tau case. More precisely, the following conditions on b are stated in [2]:

$$\limsup_{\mathbf{x}\to\mathbf{x_0}} \left| \frac{b(\mathbf{y})}{f(\mathbf{x})} \right| < \infty \quad \text{for } \mathbf{y} \in M(\mathbf{x_0}) \tag{11}$$

$$0 < \sum_{\mathbf{y} \in M(\mathbf{x_0}) \cup \{\mathbf{x_0}\}} b^2(\mathbf{y}) \tag{12}$$

These are the conditions needed for an MGM convergence proof. For the TGM, $\left|\frac{b(\mathbf{y})}{f(\mathbf{x})}\right|$ in (11) can be weakened to $\frac{b^2(\mathbf{y})}{f(\mathbf{x})}$. In [3] convergence of the TGM is proved for one- and multilevel tau and Toeplitz matrices, in [4] for multilevel circulant matrices. MGM convergence proofs for one- and multilevel tau and circulant matrices can be found in [2,1], whereas for the Toeplitz class such a proof has not yet been given.

3 Functions with whole zero curves: A Galerkin method

All structured linear systems for which multigrid methods have been developed so far correspond to nonnegative generating functions with isolated zeros. Typical applications for these types of matrices in the field of PDEs are the solution of the discrete Laplace or Poisson equation. In the following, however, we are interested in matrices corresponding to nonnegative functions having a whole zero curve instead of isolated zeros. Such linear systems arise for example when the discrete Helmholtz equation is solved with normal equations. Before working on applications we wish to develop multigrid methods for these systems in a more theoretical setting, again using generating functions for the definition of restriction and coarse grid matrices. Let us start with an example, which will be used to illustrate the main features of our multigrid methods and which will be the basis for application of our method to the discrete Helmholtz equation in an upcoming paper.

Example 1 Let $A_{mn}[f]$ be the two-level matrices belonging to a trigonometric algebra or to the Toeplitz class which correspond to the generating function

$$f(x,y) = (\rho - \cos(x) - \cos(y))^2 \qquad (0 < \rho \le 2). \tag{13}$$

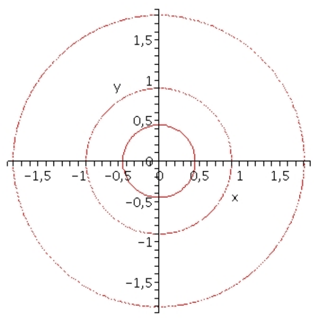

Fig. 1. Zero curves of f , f_2 , f_3 for the function $f(x,y) = (1.9 - \cos(x) - \cos(y))^2$ on the three finest levels

If the 2D-Helmholtz equation is discretized and solved, ρ takes the value $2 - \frac{k^2h^2}{2}$, where k denotes the wavenumber and h the size of a discretization step, see [6] . If $\rho = 2$, $f(x,y)$ has a single isolated zero at the origin of order 4 , and the corresponding linear systems can be solved with the method from Chapter 2. For $\rho < 2$, f is zero along a whole curve. These zero curves become larger as ρ decreases.

We wish to solve these systems with a TGM similar to the one from Section 2 . We use the same smoother and the same elementary restriction matrix E . On the matrix B , and therefore on the function b , we impose conditions similar to (11) and (12) . This time there is not a single zero, but a whole curve of zeros for which (11) and (12) must be satisfied. This means that instead of three mirror points, b must be zero at three mirror curves, which are obtained by considering the three mirror points for every point on the curve $f(x,y) = 0$. The function

$$b(x,y) = f(x+\pi,y) \cdot f(x,y+\pi) \cdot f(x+\pi,y+\pi) \tag{14}$$

in the circulant case and

$$b(x,y) = f(\pi-x,y) \cdot f(x,\pi-y) \cdot f(\pi-x,\pi-y) \tag{15}$$

in the tau and in the Toeplitz case meets these requirements. After having made this choice of b , the coarse grid function f_2 is computed with (6) and (8) . The following properties of f_2 are easily verified by direct calculation.

Lemma 1 *Let $A_{mn}[f]$ be a tau or circulant matrix whose generating function has a curve of zeros in $]-\pi/2,\pi/2[^2$. Let b be chosen with (14) or (15) , and let f_2 be computed with (6) and (8) . Then $f_2(x,y)$ has the same zero curve as $f(x/2,y/2)$, and the zeros are of the same order as the ones of $f(x,y)$. Moreover, the conditions (11) and (12) hold for each point $\mathbf{x_0}$ on the zero curve of f .*

For the construction of a Galerkin-based MGM we chose

$$b_2(x,y) = f_2(x+\pi,y) \cdot f_2(x,y+\pi) \cdot f_2(x+\pi,y+\pi) \tag{16}$$

or its tau equivalent to compute the restriction matrix on the next level. This leads to a function $f_3(x,y)$, which has the same zero curve as $f(x/4,y/4)$. Figure 1 depicts the zero curves of f ,f_2 , and f_3 for the function f from Example 1 with $\rho = 1.9$. To illustrate that our method converges fast we apply it to the matrices from Example 1 . The following table contains iteration numbers of the TGM for the two-level tau matrices corresponding to $f(x, y)$.

ρ	$n_1=(2^5-1)^2$	$n_1=(2^6-1)^2$	$n_1=(2^7-1)^2$	$n_1=(2^8-1)^2$
1.9	18	18	18	18
1.8	18	18	18	18
1.6	18	18	18	18

As in the isolated zero case, our method can also be applied to BTTB matrices. However, we either have to apply additional cutting to enforce BTTB structure of A_C , or accept that A_C is BTTB with low-rank perturbations. Nevertheless, we obtain similar numerical results if we apply our TGM to the BTTB matrices corresponding to $f(x, y)$ from Example 1 .

3.1 Two-grid convergence

We wish to prove convergence of our TGM for circulant and tau matrices by extending the convergence proofs from [4,3] . Here we carry out the proof only for tau matrices, because the proof for circulant matrices is very similar.

Theorem 2
Let $A := A_{mn}[f]$ be a two-level matrix from the circulant or tau algebra. Assume that $f(x, y)$ is a cosine nonnegative polynomial (not identically zero) with a zero curve in $]-\frac{\pi}{2}, \frac{\pi}{2}[^2$. Suppose that the smoother is the damped Richardson or Jacobi method. Furthermore, let the restriction be $R = B \cdot E$ with B from (14) or (15) and with the elementary restriction matrix E of the respective algebra. Then there exists $\gamma > 0$ such that condition (4) is satisfied, and the TGM converges.

Proof: Since the proof is similar to the proofs in [3] , we abbreviate it here. The smoothing property (3) is satisfied, because it is exactly the same as in the isolated zero case. (4) is proved by showing that it holds if we choose, for each $\mathbf{x} \in \mathbb{C}^n$, the following $\mathbf{y}$:

$$\mathbf{y} = [RR^H]^{-1} R\mathbf{x} \quad .$$

In [3] it is shown that this is true if there exists $\gamma > 0$ such that

$$I - R^H [RR^H]^{-1} R \leq \frac{\gamma}{\hat{a}}$$

with $\hat{a} = A_{j,j} > 0$. By performing a block diagonalization procedure with a permuted version of Q_{mn} from (1) this is equivalent to proving $mn - 4m_C n_C$ scalar inequalities and $m_C n_C$ 4-by-4 matrix inequalities. The scalar inequalities are of the form $\hat{a} \leq \gamma f(x_\mu, y_\nu)$ with either $\mu = m_C + 1$ or $\nu = n_C + 1$ or both. Due to our assumption that the zero curve is located within $]-\frac{\pi}{2}, \frac{\pi}{2}[^2$, f cannot vanish if x or y takes the value $\pi/2$, and the scalar inequalities hold. Because of the continuity of f and b the matrix inequalities can be reduced to a unique inequality involving 4-by-4 matrix-valued functions [3] . This inequality is of the form $L(x, y) \leq \frac{\gamma}{\hat{a}} I_4$ with

$$L(x, y) = diag(f[x, y]))^{-1/2} (I_4 - \frac{1}{\|b[x, y]\|_2^2} b[x, y](b[x, y])^T) diag(f[x, y]))^{-1/2} ,$$

where $f[x, y] = (f(\mathbf{\bar{x}_1}), f(\mathbf{\bar{x}_2}), f(\mathbf{\bar{x}_3}), f(\mathbf{\bar{x}_4}))$ with $\mathbf{\bar{x}_1} = (x, y)$ and its mirror points $\mathbf{\bar{x}_2}, \mathbf{\bar{x}_3}, \mathbf{\bar{x}_4}$. This inequality holds if $L(x, y)$ is uniformly bounded in spectral norm, which is in turn true if each element $L_{i,j}(x, y)$ of the 4-by-4 matrix function $L(x, y)$ is bounded in L^∞ . For $i \neq j$

$$L_{i,j}(x, y) = -\frac{b(\mathbf{\bar{x}_i})b(\mathbf{\bar{x}_j})}{\sqrt{f(\mathbf{\bar{x}_i})f(\mathbf{\bar{x}_j})}} \frac{1}{\|b[x, y]\|_2^2}$$

is bounded, because due to (12) one of the four terms in $b[x, y]$ is nonzero and due to (11) $\frac{b(\bar{x}_i)}{f(\bar{x}_i)}$ is bounded. For all other $\mathbf{x} \in]-\pi, \pi[^2$ which are not on the zero curve, f is strictly positive. For $i \in \{1, 2, 3, 4\}$

$$L_{i,i}(x, y) = -\frac{\sum\limits_{y \in M(\bar{x}_i)} b^2(y)}{f(\bar{x}_i)} \cdot \frac{1}{\|b[x, y]\|_2^2}$$

are also bounded, because the first factor is bounded because of (11) . ■

3.2 Problems with the Galerkin approach

The Galerkin-based TGM converges after a low number of iterations independent of the matrix size, and we have proved convergence in Theorem 2. However, the design of an MGM for practical application runs into two major problems.

- Most of the matrices we are interested in are sparse, i.e. their corresponding generating functions are trigonometric polynomials of low degree. Application of the Galerkin method with $b(x, y)$ from (14) or (15) causes matrices to become significantly denser on coarser levels. To illustrate this let us examine the matrix $A = A_{mn}[f]$ and the corresponding coarse grid matrix A_C from Example 1 . The stencil of A has 13 nonzero entries, compared to 113 of A_C . This enormous increase in density occurs in each coarsening step, and therefore makes an MGM with more than three levels inefficient. In some cases it is possible to define a function $b(x, y)$ which is less dense, but which has the same zero curves as b from (14) or (15) . For our example function f we can use $\sqrt{b(x, y)}$, resulting in a stencil of A_C which has only 41 nonzero entries. However, such a prolongation is only possible if this square root is itself a trigonometric polynomial of low degree.
- Zero curves become larger on each level. Figure 1 shows that two steps of coarsening transform a zero curve of moderate size into a considerably larger one. However, our multilevel method only works well, if the zero curve is located within the region $]-\pi/2, \pi/2[^2$. Even if the curve is close to the boundaries of this area, i.e. if for some zero (x_0, y_0) of f either x_0 or y_0 becomes greater than 1.3 or 1.4 , convergence is extremely slow. For the example function $f(x, y)$ from Example 1 we see in Figure 1 that at most three levels, i.e. two coarsening steps can be used. To illustrate this restriction, the following table shows, for the desired number m of grids in the MGM, how small ρ is allowed to be in (13) such that an m-grid method can still be applied.

$\sharp$ levels	2	3	4	5	6	7
ρ_{min}	1.25	1.80	1.95	1.987	1.997	1.9995

4 Rediscretization and approximation of the zero curve

The Galerkin method described in the previous section is the basis for the development of a fast MGM. In the following, we focus on the problem of finding coarse grid matrices which are less dense, but which do not increase the number of V-cycle iterations. This shall be achieved by using the following heuristic: *Carry out smoothing, prolongation, and restriction as before, but do not compute the coarse grid matrix A_C with a Galerkin approach.* Instead, we choose A_C corresponding to a generating function which has the same zero curve as $f(x/2, y/2)$, or which is at least a good approximation to this curve. In other words, we use a form of rediscretization on coarser grids. Finding a sparse matrix A_C whose generating function has exactly the same zero curve as $f(x/2, y/2)$ is difficult, because the function $f(x/2, y/2)$ in general corresponds to a dense matrix, and even in special cases matrices become denser on coarse grids. Therefore, we approximate $f(x/2, y/2)$ by a trigonometric polynomial of low degree.

4.1 A two-level method

In our new TGM we use the damped Richardson or Jacobi smoother and the restriction matrix $R = B \cdot E$ with E from (7) and B corresponding to b from (14) or (15). We define a function $f_2(x, y)$ in such a way that its zero curve approximates the one of $f(x/2, y/2)$. Furthermore, f_2 must be a trigonometric polynomial with only few nonzero coefficients such that the corresponding matrices are sparse. Such an approximation is carried out in two steps. First, we choose a function $f_2(x, y)$ which is similar to the original function f and which has some free parameters. In the second step, these parameters are computed such that the zero curve of f_2 shares some points with the curve $f(x/2, y/2) = 0$. A function f_2 with these two properties is expected to have a zero curve very similar to the one of $f(x/2/y/2)$. The number of free parameters should correspond to the number of points we want to fix on the curve, depending on the desired accuracy of the approximation. If f and f_2 have symmetry properties, their zero curves have additional points in common. We illustrate the construction of the coarse grid function with our example function f, which is symmetric in x and in y. The first idea for the choice of f_2 is

$$f_2^{(1)}(x, y) = [\sigma - \cos(x) - \cos(y)]^2 , \tag{17}$$

which only contains one parameter, and which results in very sparse matrices A_C. We choose σ such that the zero curves of $f(x/2, y/2)$ and $f_2^{(1)}(x, y)$ share the point $(x_1, 0)$ with $x_1 = 2\arccos(\rho - 1)$ on the positive x-axis. Then, from $f_2^{(1)}(x_1, 0) = 0$ we obtain that

$$\sigma = 1 + \cos(2\arccos(\rho - 1)) . \tag{18}$$

Because of the symmetries in f and $f_2^{(1)}$, also the points $(0, x_0)$, $(-x_0, 0)$, and $(0, -x_0)$ lie on both zero curves. For $\rho = 1.6$ the zero curves are shown in Figure 2, where the exact curve is drawn as a solid line and the approximation as a dotted line. Since this approximation is not accurate enough, we choose a slightly more complicated f_2 by adding one additional term:

$$f_2^{(2)} = [\sigma - \alpha(\cos(x) + \cos(y)) - \beta\cos(x)\cos(y)]^2 \quad . \tag{19}$$

We fix σ (e.g. by taking the value computed above) and use α and β as the free parameters. Now we determine α and β such that the zero curves of $f(x/2, y/2)$ and $f_2^{(2)}(x, y)$ have

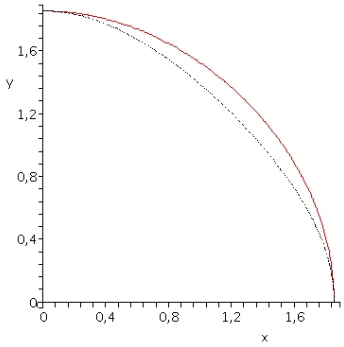

Fig. 2. Zero Curves of $f(x/2, y/2)$ and $f_2^{(1)}(x,y)$ for $\rho = 1.6$

the following two points in common: $(x_1, 0)$ and (x_2, x_2) with $x_1 = 2\arccos(\rho - 1)$ and $x_2 = 2\arccos(\rho/2)$. Due to the symmetry mentioned above the two zero curves have eight points in common. α and β are computed from $f_2^{(2)}(x_1, 0) = 0$ and $f_2^{(2)}(x_2, x_2) = 0$. The zero curve of $f_2^{(2)}(x,y)$ is an excellent approximation to the one of $f(x/2, y/2)$. Even for a large zero curve, e.g. for $\rho = 1.6$, the two curves are almost equal and cannot be destinguished from each other in Figure 2. When we use A_C corresponding to $f^{(2)}(x,y)$ in the TGM instead of the coarse grid matrix computed with the Galerkin approach, similar numerical results are obtained. For more complicated functions f with less symmetry or for extremely large matrix sizes the approximation with $f_2^{(2)}$ may not be sufficiently close. In that case we can use a third free parameter and a function of the form

$$f_2^{(3)} = [\sigma - \beta(\cos(x) + \cos(y)) - \alpha\cos(x)\cos(y) - \gamma(\cos(2x) + \cos(2y))]^2 \quad , \tag{20}$$

with fixed σ and free α, β, γ. The third common point of the two zero curves is $(2x_3, x_3)$, leading to seven more common points due to the symmetry of f. In total, the two zero curves have 16 points in common. After computing $x_3 = 2\arccos[(-1 + \sqrt{9 + 8\rho})/4]$ the parameters are obtained from $f_2^{(3)}(x_1, 0) = 0$, $f_2^{(3)}(x_2, x_2) = 0$, and $f_2^{(3)}(2x_3, x_3) = 0$. If $f_2^{(3)}$ is still not accurate enough, another term of the form $-\delta\cos(2x)\cos(2y)$ is added in (20) within the square brackets. This adds another free parameter δ and force the resulting $f_2^{(4)}$ to be zero at another point such as $(3x_4, x_4)$. If $f_2^{(4)}$ or even a trigonometric polynomial of slightly higher degree is used, the matrix A_C is not denser as the one obtained from the Galerkin approach. The true benefit of our approximation technique will become apparent when we use several levels of restriction and define an MGM. Since we use this approximation-based rediscretization approach on each level, the system matrix on coarser grids will not increase in bandwidth.

4.2 A multilevel method

We wish to extend the TGM defined in Section 4.1 to an MGM which uses a sparse coarse grid matrix obtained by rediscretization. The only restriction concerning the number of levels in the MGM lies in the increasing size of the zero curves. However if ρ is close to 2, several levels can be used as described in Section 3.2. The MGM uses a standard smoother such as damped Jacobi on each level. In the following, we explain how restriction and coarse level matrices are computed.

- On the finest level $b(x,y)$ is defined as in the Galerkin method with (14) or (15). Then, f_2 is computed as described in the previous subsection, using (19),(20), or an even better approximation.

- With this choice of f_2, $b_2(x,y)$ is computed with (16) or the tau equivalent on the second finest level. The function $f_3(x,y)$, which corresponds to the system matrix A_{CC} on the next coarser level, is computed in a similar way as f_2. f_3 is again a function of the form (19) or (20), but it must have common points with the curve $f(x/4, y/4) = 0$. With $x_1 = 4 \arccos(\rho - 1)$, $x_2 = 4 \arccos(\rho/2)$, $x_3 = 4 \arccos[(-1 + \sqrt{9 + 8\rho})/4]$ the coefficients are computed as above.
- On the next level, we compute $b_3(x,y) = f_3(x+\pi, y) \cdot f_3(x, y+\pi) \cdot f_3(x+\pi, y+\pi)$ and $f_4(x,y)$ as above. f_4 must have points in common with $f(x/8, y/8) = 0$.
- On coarser levels this procedure continues until the zero curve $f(x/2^d, y/2^d) = 0$ becomes too large, i.e. reaches the boundaries of $]-\pi/2, \pi/2[^2$.

The main advantage of this approach is that the matrices corresponding to f_2 have the same sparsity pattern as the matrices corresponding to f_3, f_4, and so on. In the following we carry out numerical tests using matrices from the circulant and tau algebras. First, we use a grid level method for matrices of circulant type. The following table contains the number of V-cycle iterations for different ρ.

ρ	$n=(2^6)^2$	$n=(2^7)^2$	$n=(2^8)^2$	$n=(2^9)^2$
1.99	15	15	15	15
1.98	16	16	16	16
1.97	18	18	18	18

For problems where ρ is close to 2 even six levels can be used for the construction of an MGM. The following table summarizes the results for matrices of the tau algebra. For the other algebras similar iteration numbers are obtained.

ρ	$n=(2^6)^2$	$n=(2^7)^2$	$n=(2^8)^2$	$n=(2^9)^2$
1.9995	15	15	15	15
1.9990	15	15	15	15
1.9985	20	20	20	20

5 Conclusions

This paper was devoted to the development of multigrid methods for linear systems of the two-level Toeplitz class and of two-level trigonometric algebras. The extension of the methods to the case where corresponding generating functions have whole zero curves faces two main difficulties: coarse-grid matrices strongly increase in bandwidth, and zero curves become too large. In this paper we have addressed the first problem by developing a multigrid method which is based on rediscretization. With a sufficiently accurate approximation of the zero curve on coarser levels our method uses coarse grid matrices whose bandwidth does not grow, no matter how many levels of restriction are used. The iteration numbers of our MGM are very similar to the ones obtained with the pure Galerkin method, and each iteration is computed significantly faster.

The second difficulty will be addressed in an upcoming paper of the authors. When the zero curve is too large for a further level of restriction and coarse grid correction, we propose to approximate it by several auxiliary problems. Combined with the results from this article, such an approach will be applied to the solution of the Helmholtz equation.

References

[1] A. Arico and M. Donatelli. A V-cycle multigrid for multilevel matrix algebras: proof of optimality and applications. *Numer. Math., to appear.*

[2] A. Arico, M. Donatelli, and S. Serra Capizzano. V-cycle optimal convergence for certain (multilevel) structured linear systems. *SIAM J. Mat. Anal. Appl.*, 26:186–214, 2004.

[3] S. Serra Capizzano. Convergence analysis of two-grid methods for elliptic Toeplitz and PDEs matrix-sequences. *Numer. Math.*, 92:433–465, 2002.

[4] S. Serra Capizzano and C. Tablino-Possio. Multigrid methods for multilevel circulant matrices. *SIAM J. Sci. Comp.*, 26(1):55–85, 2005.

[5] R. Chan, Q. Chang, and H. Sun. Multigrid methods for ill-conditioned Toeplitz systems. *SIAM J. Sci. Comp.*, 19:516–529, 1998.

[6] Y. Erlangga, C. Oosterlee, and C. Vuik. A novel multigrid based preconditioner for heterogeneous Helmholtz problems. *To appear in SIAM J. Sci. Comp.*

[7] G. Fiorentino and S. Serra. Multigrid methods for symmetric positive definite block Toeplitz matrices with nonnegative generating functions. *SIAM J. Sci. Comp.*, 17:1068–1081, 1996.

[8] U. Grenander and G. Szegö. *Toeplitz forms and their applications.* Chelsea Publishing, New York, second edition, 1984.

[9] J. W. Ruge and K. Stüben. Algebraic multigrid. In *Frontiers in Applied Mathematics: Multigrid Methods, S. McCormick Ed.*, pages 73–130, Philadelphia, 1987. SIAM.

[10] S. Serra. Preconditioning strategies for asymptotically ill-conditioned block toeplitz systems. *BIT*, 34:579–594, 1994.

[11] E. Tyrtyshnikov. Circulant preconditioners with unbounded inverses. *Linear Algebra Appl.*, 216:1–23, 1995.