Data Based Regularization Matrices for the Tikhonov-Phillips Regularization

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In Tikhonov-Phillips regularization of general form the given ill-posed linear system is replaced by a Least Squares problem including a minimization of the solution vector x, relative to a seminorm $\|Lx\|_2$ with some regularization matrix L. Based on the finite difference matrix L_k, given by a discretization of the first or second derivative, we introduce the seminorm $\|L_k D_{\tilde{x}}^{-1} x\|_2$ where the diagonal matrix $D_{\tilde{x}} := \operatorname{diag}(|\tilde{x}_1|, \ldots, |\tilde{x}_n|)$ and $\tilde{x}$ is the best available approximate solution to x.

1 Introduction

We consider the reconstruction of 1D signals $x \in \mathbb{R}^n$ occurring in discrete ill-posed problems $Hx + \eta = b$, where $\eta \in \mathbb{R}^n$ is a vector representing the unknown perturbations such as noise or measurement errors, $b \in \mathbb{R}^n$ is the observed signal, and $H \in \mathbb{R}^{n \times n}$ is the extremely ill-conditioned or even singular kernel. Our aim is to recover x as good as possible. Because of the presence of noise a regularization technique has to be applied.

Instead of $Hx + \eta = b$, the Tikhonov-Phillips regularization (TPR) in general form [1,2] solves

$$\min_x \left\{ \|Hx - b\|_2^2 + \alpha^2 \|Lx\|_2^2 \right\} \Leftrightarrow (H^T H + \alpha^2 L^T L)x = H^T b, \tag{1}$$

where $L \in \mathbb{R}^{l \times n}$, $l \leq n$ is called the regularization matrix. The regularization weight $\alpha \geq 0$ is to be chosen such that both minimization criterions yield the optimal value together: the computed solution x should be as close as possible to the original problem and sufficiently regular. Instead of using the 2-norm as a means to control the error in the solution and to obtain regularity, i.e., instead of using $L = I$ leading to standard TPR, another possibility is to use discrete smoothing norms $\|L_k x\|_2$, where L_k is an approximation to the first or second derivative operator [1,2], that is $L_1 := \operatorname{upperbidiag}(1, -1) \in \mathbb{R}^{(n-1) \times n}$ and $L_2 := \operatorname{uppertridiag}(-1, 2, -1) \in \mathbb{R}^{(n-2) \times n}$. Smoothing norms will improve the reconstruction if the underlying signal is smooth because the penalty term $\|L_k x\|_2$ will have no effect on the smooth signal part but provoke large seminorms on the oscillating part corresponding to noise. Observing the exemplary smooth vector $(1, 1, \ldots, 1)^T$ in the noise-free case, a smoothing norm has no effect on the signal part as $L_k(1, 1, \ldots, 1)^T = (0, 0, \ldots, 0)^T$. Consequently, rough oscillations caused by noisy components can be suppressed during the reconstruction and the regularized approximation will satisfy inherent smoothness properties.

2 Data Based Regularization Matrices and Smoothing Norms

In many cases the solution vector is not smooth and therefore smoothing seminorms will spoil the reconstruction. Here, one can take into account the solution vector respectively the best available approximation $\tilde{x}$. This was considered for the first time in [4] in the context of preconditioning of linear solvers for ill-posed problems. An improved reconstruction was observed by applying tridiagonal preconditioners which act like $\operatorname{tridiag}(1, 2, 1)$ near continuous components but like $\operatorname{tridiag}(0, 1, 0)$ near discontinuities. Based on this, incorporating the signal data in form of $D_{\tilde{x}}$ as a preconditioner for direct and iterative regularization methods was proposed in [5]. For weakly blurred signals containing discontinuities and zero components, this approach revealed strong improvement. Note that in [8] a seminorm for TPR is proposed which is constructed by solving $\min_L \|L\tilde{x}\|_2$ for a prescribed structure of L.

Here, we propose a combination of data based regularization with differential operators in (1) by defining the penalty term

$$\|L_k D_{\tilde{x}}^{-1} x\|_2, \quad \text{where} \quad D_{\tilde{x}} := \operatorname{diag}(|\tilde{x}_1|, \ldots, |\tilde{x}_n|) \tag{2}$$

and $\tilde{x}$ is the best approximation constructed via TPR using $L = I$ or $L = L_k$, denoted as $\tilde{x}_I$ and $\tilde{x}_{L_k}$, respectively. In the case that $|\tilde{x}_i| < \epsilon$ we set $(D_{\tilde{x}})_{ii} = \epsilon$ with $0 < \epsilon \ll 1$. The seminorm satisfies the condition of acting only on the noise subspace leaving the solution's signal part unchanged. Similar to smoothing norms, we observe for $k = 1, 2$ that $L_k \left(D_{\tilde{x}}^{-1} \tilde{x} \right) = L_k(1, 1, \ldots, 1)^T = (0, 0, \ldots, 0)^T$. Following [5], we may apply an iterative process of constructing $D_{\tilde{x},s}$ based on an available reconstruction $\tilde{x}^{(s-1)}$. I.e., we construct $D_{\tilde{x},1}$ from the best available solution, e.g., $\tilde{x}^{(0)} = \tilde{x}_I$ or $\tilde{x}^{(0)} = \tilde{x}_{L_k}$, use (2) in TPR to obtain the improved signal $\tilde{x}^{(1)}$, construct $D_{\tilde{x},2}$, and so forth. In most cases $s = 3$ is sufficient.

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3 Results and Conclusions

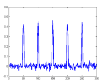
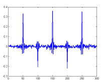
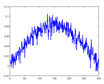
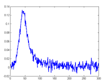
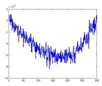
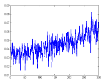
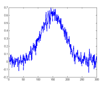
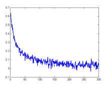
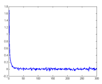
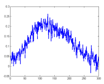
We focus on ill-posed problems from Regularization Tools [3] which provides discretizations H of the integral operators occurring in Fredholm integral equations of the first kind. We assume that the entries of η are random numbers resulting from the same Gaussian distribution with zero mean and standard deviation $\xi \in \mathbb{R}$. Hence, we use white noise of different magnitude and perform all computations on normalized values. Concerning α in the TPR, we illustrate the maximum obtainable improvement for a perfect estimator. We compute it via the MATLAB [7] function `fminbnd` which attempts to find a local minimizer in our chosen interval $[0, 5000]$. We refine the search by setting the termination tolerance to `TolX` = 10^{-9}. Hence, we measure the quality of the reconstructions via the relative error $\|x - \tilde{x}\|_2 / \|x\|_2$. All our test problems have a fixed problem size $n = 300$. Note that for the `prolate` problem from [7] we use two different signals: a pulse sequence with positive and negative discontinuities and a smooth sine arch.

Table 1 Problems from REGULARIZATION TOOLS [3] and MATLAB [7] used in Table 2.

No.	Problem
1	`blur1D(300,3,2)`
2	`gallery('prolate',300,0.34)`
3	`gallery('prolate',300,0.34)`
4	`heat(300,5)`
5	`deriv2(300,1)`
6	`foxgood(300)`
7	`phillips(300)`
8	`i_laplace(300,1)`
9	`i_laplace(300,2)`
10	`shaw(300)`

Following Table 2, we obtain significant improvement for weakly perturbed signals containing discontinuities and zero components when using $L = D^{-1}_{\tilde{x},s}$ (cf. Problem 1 and 2). For smooth signals, where discrete smoothing norms yield significant improvement, the usage of $L = L_k D^{-1}_{\tilde{x},s}$ may additionally improve the reconstruction (see Problems 3–7). Note that there are problems such as Problem 10 where no improvement can be obtained by using any of the considered seminorms. In [6], we proposed seminorms $\|L_H x\|_2$ which incorporate the spectral data of the operator H. A promising subject to future research is the combination of this approach with data based regularization in TPR.

Table 2 Relative error $\|x - \tilde{x}\|_2 / \|x\|_2$ for problems listed in Table 1. The exact solution is x while $\tilde{x}$ denotes the reconstruction.

Regul. Matrix	No.	$Hx+\eta$	Noise $\xi = 10\%$	$\xi = 1\%$	$\xi = 0.1\%$	No.	$Hx+\eta$	Noise $\xi = 10\%$	$\xi = 1\%$	$\xi = 0.1\%$
I	1		**0.9165**	**0.5576**	**0.1850**	2		**0.8956**	**0.5786**	**0.5638**
L_1			0.9387	0.5668	0.1896			0.9624	0.5783	0.5637
$L_1 D^{-1}_{\tilde{x},3}$			0.7029	0.1742	0.0258			0.8200	0.0810	0.0082
$D^{-1}_{\tilde{x},3}$			0.3937	0.0159	0.0015			0.5652	0.0285	0.0028
I	3		**0.8204**	**0.1485**	**0.0176**	4		**0.6041**	**0.2042**	**0.0540**
L_2			0.1608	0.0208	0.0027			0.3545	0.0771	0.0172
$L_2 D^{-1}_{\tilde{x},3}$			0.1539	0.0209	0.0027			0.2224	0.0400	0.0100
I	5		**0.5625**	**0.4343**	**0.2480**	6		**0.1097**	**0.0276**	**0.0074**
L_2			0.0853	0.0085	0.0009			0.0980	0.0098	0.0010
$L_2 D^{-1}_{\tilde{x},3}$			0.0593	0.0059	0.0006			0.0876	0.0088	0.0009
I	7		**0.2610**	**0.0811**	**0.0271**	8		**0.2937**	**0.2127**	**0.1724**
L_1			0.2401	0.0777	0.0266			0.3190	0.6478	0.6529
$L_1 D^{-1}_{\tilde{x},3}$			0.2104	0.0650	0.0268			0.1403	0.0282	0.0085
I	9		**0.8297**	**0.8116**	**0.7883**	10		**0.2195**	**0.1584**	**0.0990**
L_2			0.1491	0.0954	0.0917			0.5713	0.5401	0.5381
$L_2 D^{-1}_{\tilde{x},3}$			0.1576	0.1302	0.1309			0.5819	0.5484	0.5459

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