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On cylindrical tensegrity structures

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Abstract
We study cylindrical tensegrity structures, such as triplex, quadruplex and higher order n-plexes. Most high order n-plexes possess novel configurations that have the same connectivity or number of nodes but differ in their overall geometric arrangement. We present a closed-form solution for all these cylindrical structures. Moreover, we enumerate, classify, and investigate their structural properties.

1 Introduction

Tensegrity structures (e.g. Motro [7]) are usually defined as a discontinuous set of compressional members inside a continuous network of tensile members. They are mechanically stabilized by the action of pre-stress and are self-equilibrated without the application of an external force. The analysis of tensegrity structures requires an initial form-finding, e.g. Tibert et al. [11]. Exact solutions exist for single configurations of cylindrical tensegrity structures, such as triplex, quadruplex and higher order n-plexes, Motro [7], Tibert et al. [11], Connelly et al. [3]. However, most cylindrical tensegrities have a certain number of topologically equivalent configurations. They have the same connectivity or same number of nodes but differ in their geometric arrangement and have not been systematically investigated yet.

In this work we present a closed-form solution for the form-finding of the equivalent configurations for all n-plexes. We describe and classify them in the first part of the paper and discuss their form-finding and structural properties in the following parts.

2 Classification of tensegrity cylinders

A cylindrical tensegrity structure consists of two polygons, similar up to an affine transformation, which are connected by a set of n bracing cables and n struts. Each polygon consists of $n \geq 3$ nodes interconnected by n cables. The following parameters define the standard representation for n-plexes:

(i) The number $j = 1$ of steps between the nodes that are connected by a strut. (ii) The number $k = 1$ of steps between consecutively connected nodes along the convex hull of a given polygon.

For example, figure 1a depicts a 5-plex where a strut (solid line $\overline{AB'}$) connects nodes A and B', and B' is $j = 1$ steps apart from node A', i.e. the node corresponding to A in the "top" polygon. Notice

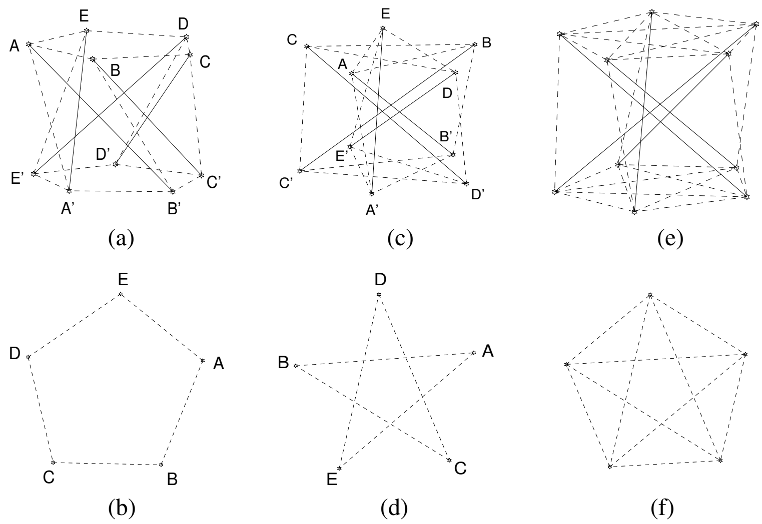

Figure 1: Example of a 5-plex with the standard connectivity $j = k = 1$: (a), (c) the two possible geometrical configurations with $p = 1$ and $p = 2$ respectively; (b), (d) constituent polygons of (a) and (c), respectively; (e), (f) structures corresponding to (c) and (d) with a stabilizing ring.

that the sides of these cylinders are fully triangulated. Additionally, figure 1b depicts the constitutive polygon of a 5-plex with $k = 1$, i.e. each node is connected to its immediate neighbour in the plane.

We study different configurations of these n-plexes that have the same connectivity $j = k = 1$ but that differ in the *geometric arrangement* of the nodes. We define p as the number of steps between geometric consecutive nodes along the convex hull of the polygons. Figure 1c depicts the alternative geometry of the 5-plex with the same node-to-node connections as in 1a, but with $p = 2$. It exhibits a typical star-like shape as opposed to the case $p = 1$.

3 Form-finding

There are several ways to calculate the equilibrium configuration of n-plexes, see Tibert et al. [11] or Motro [7] for a survey. Henceforth, we use the same standard terminology. For instance, it is possible to find the equilibrium configuration (i) by a ratio between strut and cable lengths; (ii) by a twisting angle between the constitutive polygons; or (iii) by calculating tension coefficients[1].

We present a closed-form solution for the form-finding of any n-plex in terms of tension coefficients. This simplifies the procedure presented by Vassart [12], who described a semi-numerical form-finding procedure based upon the force density method. Feasible tension coefficients are found when the matrix:

$$S = C^T \,\mathrm{diag}(u)\, C \tag{1}$$

has $nullity(S) = \mathcal{D} + 1 = 4$, e.g. Motro [7], Connelly et al. [2], Graver et al. [4]. Here $C \in \Re^{b \times l}$ is a connectivity matrix of a structure in $\mathcal{D} = 3$ dimensions, with b bars and l nodes; $u \in \Re^b$ is a vector of tension coefficients some of which are unknown; $S \in \Re^{l \times l}$ is a force density matrix (FDM); and *diag* is a diagonal matrix.

The nullity is fulfilled by (i) reducing S (eq. 1) to a row echelon form, (ii) looking for a polynomial such that the nullity is 4, and (iii) solving for its positive real roots. The positive real roots q are the tension coefficients that self-equilibrate the structure. Although this form-finding procedure is sound and complete, it is very time-consuming. The first three columns of Table 1 show the order, the polynomial that contains the roots (i.e. minimal polynomial) and their positive real roots q for some n-plexes.

[1] Also known as force density coefficients.

n	minimal polynomial	positive real root(s)	closed form
3	q^2-3	$\sqrt{3}$	$2\sin(\pi/3)$
4	q^2-4	$\sqrt{2}$	$2\sin(\pi/4)$
5	q^4-5q^2+5	$\frac{\sqrt{10\pm2\sqrt{5}}}{2}$	$2\sin(\pi/5), 2\sin(2\pi/5)$
6	$q-1$	1	$2\sin(\pi/6)$
7	$q^6-7\,q^4+14\,q^2-7$	$0.8677, 1.5636, 1.9498$	$2\sin(\pi/7), 2\sin(2\pi/7), 2\sin(3\pi/7)$
8	$q^4-4\,q^2+2$	$\sqrt{2\pm\sqrt{2}}$	$2\sin(\pi/8), 2\sin(3\pi/8)$
9	$q^6-6\,q^4+9\,q^2-3$	$0.6841, 1.2855, 1.9696$	$2\sin(\pi/9), 2\sin(2\pi/9), 2\sin(4\pi/9)$
10	$\left(q^2+q-1\right)\left(q^2-q-1\right)$	$\frac{\sqrt{5}}{2}\pm\frac{1}{2}$	$2\sin(\pi/10), 2\sin(3\pi/10)$

Table 1: Solutions of some n-plexes in terms of a tension coefficient q.

3.1 Closed-form solution

Trigonometric functions of angles $(k\pi/n)$ where k and n produce constructible polygons are algebraic numbers, e.g. Ribenboim [9]. Thus, they correspond to real roots of minimal polynomials with integer coefficients. Consequently, we can compute the positive real roots q directly *without* calculating any minimal polynomial.

Constructible polygons are created when the rational number (p/n) is formed from relative primes or co-primes, i.e. $GCD(n,p) \equiv 1$. They are visualized as single looped curves with $(p-1)n$ crossings, see Figures 1b and 1d. By simple inspection we see that the function *sine* generates the solutions of all configurations of n-plexes. These solutions are given in the fourth column of Table 1. We write the closed-form solution[2] of tension coefficients q as follows:

$$q = 2\sin\left(\frac{p}{n}\pi\right) \qquad ; n \geq 3 \text{ and } p \in \{i|GCD(n,i) \equiv 1\} \tag{2}$$

The number of real roots (or algebraic degree) of the trigonometric function $sin(\pi/n)$ is given in terms of the Euler's totient function, $\phi(n)/2$, see e.g. Beslin et al. [1], Ribenboim [9]. Notice that by definition $GCD(n,1) \equiv 1, \forall n$, thus at least one cylinder exists for any n. Interestingly, there are only three n-plexes $(n = 3, 4, 6)$ with $\phi(n)/2 \equiv 1$. They have no equivalent star-like configurations. Moreover, it follows from the equation 2 that the hexaplex is the only n-plex with unitary tension coefficients.

4 Structural properties

The number of independent states of self-stress is found to be $s = 1$, regardless of p. The number of inextensional modes of deformation (mechanisms) is $m = 2n - 5$. Tarnai [10] and Murakami [8] reported similar states of self-stress in their study of n-plexes with $p = 1$. Our study shows, however, that this result also holds for n-plexes with $p > 1$.

4.1 Initial stiffness

The vector of tension coefficients contributes to the tangent stiffness matrix of pre-stressed, kinematically indeterminate structures, see e.g. Guest [5], Murakami [8]. We calculate the eigenvalues of the tangent stiffness matrix of n-plexes subjected to inextensional displacements and zero external load. This initial response only affects the geometric stiffness, e.g. Guest [5], and proves to be dependent on p. Since the FDM (with tension coefficients from eq. 2) have $2(p-1)$ negative eigenvalues, the tangent stiffness matrix has at least that number of negative eigenvalues as well. This fact proves that structures with

[2]Interestingly, the geometric representation of q is the length between any two consecutive nodes at p steps apart along the convex hull of a given polygon; see length c in Hinrichs [6], pp. 11. Moreover, Connelly et al. [3] derived to a similar expression for q, called "stress", in their proof of lemma 3 (set $j = k$ and unitary radii γ).

$p = 1$ are the only stable equilibrium configuration. For instance, figure 1c has a state of self-stress but its equilibrium configuration is unstable.

4.2 Stabilization of star-like cylinders

Configurations with $p > 1$ can be stabilized by adding extra cables. This is done by connecting the arms of star-like structures such as to form rings enclosing the stars. The new structures are still tensegrity structures. For example, the ringed version of Figure 1c and 1d are shown in Figure 1e and 1f. Star-like structures with rings are found to have a positive tangent stiffness, $s = 7$ and *only* one mechanism, $m = 1$, for n odd.

5 Conclusions

We have introduced and classified novel tensegrity structures that are topologically equivalent to the cylindrical tensegrity structures (n-plexes) as they are known in the literature. We have also presented an efficient and new form-finding procedure for all topologically equivalent configurations of any n-plex in terms of a trigonometric function. The number of such configurations is given in terms of Euler's totient function. A structural analysis of the star-like configurations shows that they have a negative tangent stiffness but that they can be stabilized by constructing rings of tension cables around them.

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