HIGH TEA @ SCIENCE

Random Ordinary Differential Equations (RODE)

F. Menhorn, T. Neckel, A. Parra, C. Riesinger, F. Rupp

Technische Universität München

March 14, 2016

Motivation

- goal: efficient tools for numerical simulation of random effects

Motivation

- goal: efficient tools for numerical simulation of random effects
- RODE as an alternative setting for SODE
  $\Rightarrow$ massive simulations of pathes

Motivation

- goal: efficient tools for numerical simulation of random effects
- RODE as an alternative setting for SODE
  $\Rightarrow$ massive simulations of pathes
- approach:
  1. HPC aspects of RODE simulations: parallelisation
  2. math aspects: relation of RODE to classical UQ

Outline

Motivation

Random Differential Equations (RODEs)

Definition of RODEs
Properties of RODEs
Application Example: Kanai-Tajimi Earthquake Model

Solving RODEs on GPU clusters

Definition of RODEs

Assumptions

- $X : I \times \Omega \to \mathbb{R}^m$: stochastic process (continuous sample paths)
- $f : \mathbb{R}^d \times I \times \Omega \to \mathbb{R}^d$ continuous

Definition

RODE on $\mathbb{R}^d$

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = f(X_t(\cdot), t, \omega), \qquad X_t(\cdot) \in \mathbb{R}^d \tag{1}$$

is a non-autonomous ordinary differential equation

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = F_\omega(x, t), \qquad x := X_t(\omega) \in \mathbb{R}^d \tag{2}$$

for almost all pathes $\omega \in \Omega$.
[Bun72]

$\Rightarrow$ **path-wise solutions!**

RODE vs. SODE

SODE

$$dX_t = \underbrace{a(X_t, t)}_{\text{drift}}\, dt + \underbrace{b(X_t, t)}_{\text{diffusion}}\, dW_t \,.$$

typically: considerable mathematical effort (Itô's calculus etc.)

RODE vs. SODE

SODE

$$\mathrm{d}X_t = \underbrace{a(X_t, t)}_{\text{drift}}\,\mathrm{d}t + \underbrace{b(X_t, t)}_{\text{diffusion}}\,\mathrm{d}W_t\,.$$

typically: considerable mathematical effort (Itô's calculus etc.)

Doss-Sussmann/Imkeller-Schmalfuss correspondence (DSIS)

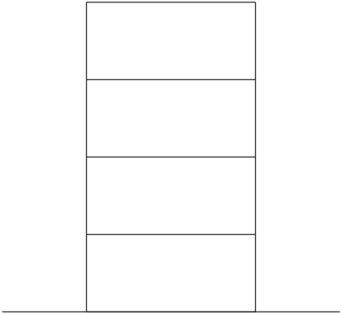
Under sufficient regularity conditions [IS01]:

SODEs and RODEs are conjugate!

$\Rightarrow$ rewrite SODE as RODE by changing the driving stochastic process:
typically: white noise $\rightsquigarrow$ stationary *Ornstein-Uhlenbeck* (OU) process

Application Example: Kanai-Tajimi Earthquake Model

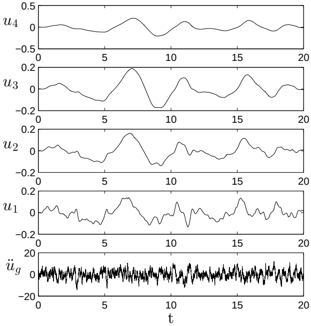
4-storey wireframe building



$$\ddot{u}_g \quad \Longleftrightarrow$$

$$\ddot{u} + C\dot{u} + Ku = F(t),$$

Kanai-Tajimi excitation & storey displacements



Kanai-Tajimi Earthquake Model as RODE

SODE

$$\ddot{u}_g = \ddot{x}_g + \xi_t = -2\zeta_g\omega_g\dot{x}_g - \omega_g^2 x_g$$

$$\ddot{x}_g + 2\zeta_g\omega_g\dot{x}_g + \omega_g^2 x_g = -\xi_t, \qquad x_g(0) = \dot{x}_g(0) = 0$$

ξ_t: white noise

ζ_g, ω_g: model parameters (e.g. $\zeta_g = 0.64$, $\omega_g = 15.56[rad/s]$)

Kanai-Tajimi Earthquake Model as RODE

SODE

$$\ddot{u}_g = \ddot{x}_g + \xi_t = -2\zeta_g\omega_g\dot{x}_g - \omega_g^2 x_g$$

$$\ddot{x}_g + 2\zeta_g\omega_g\dot{x}_g + \omega_g^2 x_g = -\xi_t, \qquad x_g(0) = \dot{x}_g(0) = 0$$

ξ_t: white noise

ζ_g, ω_g: model parameters (e.g. $\zeta_g = 0.64$, $\omega_g = 15.56[rad/s]$)

SODE in vector form

$$\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -2\zeta_g\omega_g y + \omega_g^2 x \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{d}W_t$$

Kanai-Tajimi Earthquake Model as RODE

SODE

$$\ddot{u}_g = \ddot{x}_g + \xi_t = -2\zeta_g\omega_g\dot{x}_g - \omega_g^2 x_g$$

$$\ddot{x}_g + 2\zeta_g\omega_g\dot{x}_g + \omega_g^2 x_g = -\xi_t, \qquad x_g(0) = \dot{x}_g(0) = 0$$

ξ_t: white noise

ζ_g, ω_g: model parameters (e.g. $\zeta_g = 0.64$, $\omega_g = 15.56[rad/s]$)

SODE in vector form

$$\mathrm{d}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -2\zeta_g\omega_g y + \omega_g^2 x \end{pmatrix}\mathrm{d}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathrm{d}W_t$$

RODE (DSIS correspondence)

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -(z_2 + O_t) \\ -2\zeta_g\omega_g(z_2 + O_t) + \omega_g^2 z_1 + O_t \end{pmatrix}$$

Kanai-Tajimi Earthquake Model: OU Process

Definition of Ornstein-Uhlenbeck process:

$$dO_t = \theta(\mu - O_t)dt + \sigma dW_t$$

Kanai-Tajimi Earthquake Model: OU Process

Definition of Ornstein-Uhlenbeck process:

$$dO_t = \theta(\mu - O_t)dt + \sigma dW_t$$

Numerical scheme (exact integration [Gil96])

$$O_{t+h} = \mu_x O_t + \sigma_x n_1$$

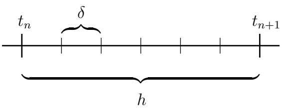
with

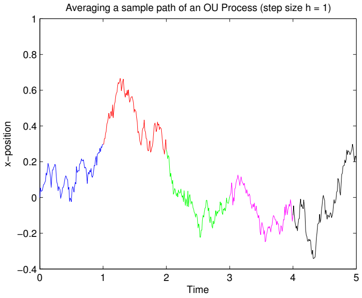
- h: timestep size
- $\mu_x = e^{-h \cdot \theta}$
- $\sigma_x^2 := \frac{\sigma^2}{2\theta}(1 - e^{-h \cdot 2\theta})$
- n_1: sample value of $\mathcal{N}(0, 1)$

Numerical Schemes for RODEs

Specific methods to achieve order of convergence [GK01, KJ07, NR13]:

- Averaged Euler
- Averaged Heun
- *K*-RODE Taylor schemes





Numerical Schemes for RODEs II

For an RODE in the following form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = G(t) + g(t)H(X),$$

use averaged values for $g(t)$ on δ:

$$\bar{g}^{(1)}_{h,\delta}(t) = \frac{1}{N}\sum_{j=0}^{N-1} g(t + j\delta),$$

Numerical Schemes for RODEs II

For an RODE in the following form

$$\frac{\mathrm{d}X}{\mathrm{d}t} = G(t) + g(t)H(X),$$

use averaged values for $g(t)$ on δ:

$$\bar{g}_{h,\delta}^{(1)}(t) = \frac{1}{N} \sum_{j=0}^{N-1} g(t + j\delta),$$

$\Longrightarrow$ *averaged Euler scheme* (timestep h for $x_n \to x_{n+1}$)
same averaging procedure for $G(t)$:

$$\begin{aligned} x_{n+1} &= \frac{1}{N} \sum_{j=0}^{N-1} \{x_n + hG(t_n + j\delta) + hg(t_n + j\delta)H(x_n)\} \\ &= x_n + \frac{1}{N} \sum_{j=0}^{N-1} hG(t_n + j\delta) + \frac{H(x_n)}{N} \sum_{j=0}^{N-1} hg(t_n + j\delta)\ . \end{aligned}$$

$\Longrightarrow$ *global* discretisation error of order 1

Solving RODEs on GPU clusters

and now let's switch to the dark side: Computer Science ;-)

http://bgr.com/2015/11/19/darth-vader-daily-life-photos/

Literature I

H. Bunke.
Gewöhnliche Differentialgleichungen mit zufälligen Parametern.
Akademie-Verlag, Berlin, 1972.

D. T. Gillespie.
Exact numerical simulation of the ornstein-uhlenbeck process and its integral.
Physical Review E, 54(2):2084–2091, 1996.

L. Grüne and P. E. Kloeden.
Pathwise approximation of random ordinary differential equations.
BIT Numerical Mathematics, 41(4):711–721, 2001.

P. Imkeller and B. Schmalfuss.
The conjugacy of stochastic and random differential equations and the existence of global attractors.
Journal of Dynamics and Differential Equations, 13(2):215–249, 2001.

Literature II

P. Kloeden and A. Jentzen.
Pathwise convergent higher order numerical schemes for random ordinary differential equations.
Proc. R. Soc. A, 463:2929–2944, 2007.

T. Neckel and F. Rupp.
Random Differential Equations in Scientific Computing.
Versita, De Gruyter publishing group, Warsaw, 2013.
open access.