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Optimal cubature formulas for Fourier coefficients of periodic functions in Sobolev spaces

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Summary. The purpose of the paper is to discuss methods for constructing weighted cubature formulas in Sobolev spaces with periodic members. These formulas are intended to approximate the calculation of Fourier coefficients of functions under consideration. The explicit formulas for the weights and the errors of optimal cubature formulas are obtained by techniques from functional analysis and partial differential equations.

Key words: Fourier coefficients, optimal cubature formulas, difference operators of Sobolev type, the error estimations, the Filon quadrature formula

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1. Introduction and statement of the results

Let $x = (x_1, \dots, x_n) \in R^n$, H be a real square $n \times n$ matrix, and $\det H > 0$. To the matrix H we assign its *fundamental parallelepiped* Ω_0 by putting

$$\Omega_0 = \{x \in R^n : x = Hy,\ 0 \leq y_j < 1,\ j = 1, 2, \dots, n\}.$$

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We consider the problem of computing the Fourier coefficients

$$a_\varphi[\gamma] = \int\limits_{\Omega_0} e^{i2\pi H^{*-1}\gamma x}\varphi(x)\,dx$$

of an (unknown) continuous real function $\varphi(x)$. Here H^* is the adjoint matrix to H; H^{*-1} is the inverse matrix to H^*; $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is an integer column-vector (i.e., having integer entries), $\gamma \neq 0$; and $H^{*-1}\gamma x$ denotes the inner product of the vector $H^{*-1}\gamma$ and the vector x. If H is an orthogonal matrix, then $H^{*-1} = H$. The domain Ω_0 of integration and the weight function $e^{i2\pi H^{*-1}\gamma x}$ are related by the equality

$$\int\limits_{\Omega_0} e^{i2\pi H^{*-1}\gamma x}\,dx = \int\limits_{Q} e^{i2\pi\gamma y}\,dy = \prod_{j=1}^{n}\int\limits_{0}^{1} e^{i2\pi\gamma_j y_j}\,dy_j = 0,$$

with Q the unit cube in R^n.

Given a positive integer M, we consider linear rules of the type

$$Q[\varphi] = \sum_{hH\beta\in\Omega_0} c[\beta]\varphi(hH\beta), \quad h = 1/M,$$

for the approximation of $a_\varphi[\gamma]$. Naturally, we assume that the values $\varphi(hH\beta)$ with $hH\beta \in \Omega_0$ are known. The sum $Q[\varphi]$ has N terms; $N = 1/h^n$; and N is called the number of formula nodes.

To each rule $Q[\varphi]$ we assign the error

$$(l,\varphi) \equiv \int\limits_{\Omega_0} e^{i2\pi H^{*-1}\gamma x}\varphi(x)\,dx - \sum_{hH\beta\in\Omega_0} c[\beta]\varphi(hH\beta), \tag{1.1}$$

with $\varphi(x)$ a member of some Hilbert space $X = X(\Omega_0)$. The value of $l(x)$ is determined at every function $\varphi(x)$ in X only in the case when $\varphi(x)$ is continuous. We therefore naturally require that X be embedded in $C(\overline{\Omega}_0)$, the space of functions that are continuous on $\overline{\Omega}_0$. Moreover, we require that the embedding of X to $C(\overline{\Omega}_0)$ be compact. Consequently, the error $l(x)$ is a bounded linear functional on X; i.e., $l(x) \in X^*$, the dual space.

We define

$$\rho(Q, X, \gamma, N) = ||l \mid X^*|| = \sup_{||\varphi|X||\leq 1}\{|a_\varphi[\gamma] - Q[\varphi]|\}$$

and have the error bounds

$$|a_\varphi[\gamma] - Q[\varphi]| \leq \rho(Q, X, \gamma, N)||\varphi \mid X||.$$

Clearly, $\rho(Q,X,\gamma,N)$ is a measure of the quality of the rule Q; and the following number is of the special interest:

$$\rho^{opt}(X,\gamma,N) = \inf_Q \rho(Q,X,\gamma,N),$$

where the inf is taken over all considered $Q[\varphi]$ (that means over all weights $\{c[\beta] \mid hH\beta \in \Omega_0\}$). Following Brass (1991) we look at $\rho^{opt}(X,\gamma,N)$ as a measure for the costs (or the complexity) required for the computation of a Fourier coefficient of a function in X.

The rule Q^{opt} such that $\rho(Q^{opt},X,\gamma,N) = \rho^{opt}(X,\gamma,N)$ is called an X-optimal cubature formula for the computation of a Fourier coefficient $a_\varphi[\cdot]$. We now turn to the following

Problem 1. *Given a number of nodes N, find the X-optimal cubature formula of the form (1.1).*

In this paper, we solve Problem 1 in case of Sobolev spaces with periodic members. The exact definitions are as follows.

Let the function $\varphi(x)$ with domain R^n have all derivatives up to order k locally integrable in R^n with the integral

$$\int_\Omega \sum_{|\alpha|=k} \frac{k!}{\alpha!} |D^\alpha \varphi(x)|^2 \, dx = ||\varphi \mid L_2^{(k)}(\Omega)||^2$$

finite for every bounded domain Ω. Also assume that $2k > n$, and $\varphi(x)$ is a periodic function with periodic matrix H, i.e.,

$$\varphi(x + H\beta) = \varphi(x), \quad \forall \beta \in Z^n. \tag{1.2}$$

Here and in the sequel Z denotes the set of integers. Thus, β is an integer column-vector.

We denote the Sobolev space composed by the function with prescribed properties by $\widetilde{W}_2^{(k)}(H)$ (cf. Sobolev and Vaskevich (1997)).

Given a positive integer $m > n/2$ and the numbers a_k, $k = 0, 1, \ldots$, such that

$$0 < a_0 < +\infty, \ a_1 = a_2 = \ldots = a_{m-1} = +\infty,$$

$$0 < a_k < +\infty \text{ for } k \geq m,$$

we consider the collection of all functions $\varphi = \varphi(x)$ such that φ is a member of $\widetilde{W}_2^{(k)}(H)$ for $\forall k \geq m$, and

$$\sum_{k=0}^{\infty} \frac{||\varphi \mid L_2^{(k)}(\Omega_0)||^2}{a_k^2} < +\infty.$$

We denote this set by $\widetilde{W}_2^{(a_k)}(H)$. The norm of $\varphi \in \widetilde{W}_2^{(a_k)}(H)$ is defined as follows

$$||\varphi \mid \widetilde{W}_2^{(a_k)}(H)|| = \sum_{k=0}^{\infty} \frac{||\varphi \mid L_2^{(k)}(\Omega_0)||^2}{a_k^2}.$$

The space $\widetilde{W}_2^{(a_k)}(H)$ is called the periodic Sobolev space of infinite order (cf. Dubinskiĭ (1980)). Fixing the sequence (a_k), we will write $\widetilde{X}(H)$, or simply $\widetilde{X}$, instead of $\widetilde{W}_2^{(a_k)}(H)$.

Each function from the set

$$\{e^{i2\pi H^{*-1}\beta x} \mid \beta \in Z^n\} \tag{1.3}$$

satisfies (1.2). We assume that all exponents from (1.3) are the members of $\widetilde{X}(H)$. Hence, we have

$$A_0[\beta] \equiv \sum_{k=0}^{\infty} \frac{(2\pi |H^{*-1}\beta|)^{2k}}{a_k^2} < +\infty, \quad \forall \beta \in Z^n, \tag{1.4}$$

with $|H^{*-1}\beta|$ the Euclidean norm of the vector $H^{*-1}\beta$. Moreover, the equality holds

$$||\varphi \mid \widetilde{X}(H)||^2 = \sum_{k=0}^{\infty} \frac{||\varphi \mid L_2^{(k)}(\Omega_0)||^2}{a_k^2} = \frac{1}{|H|} \sum_{\beta} A_0[\beta] |a_\varphi[\beta]|^2,$$

with $\varphi(x)$ a member of $\widetilde{X}(H)$ and $a_\varphi[\beta]$ a Fourier coefficient of φ. The space $\widetilde{W}_2^{(a_k)}(H)$ is a Hilbert space with orthogonal basis (1.3). If $a_k \to +\infty$ for $k = m+1, m+2, \ldots$, then $\widetilde{X}(H)$ change into the Sobolev space $\widetilde{W}_2^{(m)}(H)$ of finite order. If $a_0 \to +\infty$, $a_m = 1$, and $a_k \to +\infty$ for $k = m+1, m+2, \ldots$, then $\widetilde{X}(H)$ change into the Sobolev space $\widetilde{L}_2^{(m)}(H)$ (see, e.g., Sobolev and Vaskevich (1997)).

Theorem 1. *There exists a unique $\widetilde{X}(H)$-optimal cubature formula of the form (1.1) with the given number of nodes $N = h^{-n}$. The weights of this cubature formula and the entire function $A_0[\beta]$ defined by (1.4) are related by*

$$c_0^{(\gamma)}[\beta] = h^n |H| \left[1 + \sum_{\alpha \neq 0} \frac{A_0[\gamma]}{A_0[\gamma - \alpha/h]}\right]^{-1} e^{i2\pi h\beta\gamma}, \tag{1.5}$$

with α, β *the members*[1] *of* Z^n *and* $hH\beta \in \Omega_0$. *The series in (1.5) converges absolutely. The norm of the optimal error* $l_0^{(\gamma)}(x)$ *depends on* γ, N, H, *and* (a_k) *as follows*

(1.6)

$$||l_0^{(\gamma)} \mid \widetilde{X}^*(H)||^2 = |H|\Big(\sum_{\beta\neq 0} \frac{1}{A_0[\gamma-\beta/h]}\Big)\left[1+\sum_{\beta\neq 0}\frac{A_0[\gamma]}{A_0[\gamma-\beta/h]}\right]^{-1}.$$

Theorem 1 was presented in Bulgak and Vaskevich (2000) without the proof. If $a_k \to +\infty$ for $k = m+1, m+2, \ldots$, then (1.4)–(1.6) change into more simple equalities; and we can use them for the members of the Sobolev spaces with the finite smoothness.

Let $\widetilde{X}(H) = \widetilde{L}_2^{(m)}(H)$, i.e., $A_0[\beta] = (2\pi|H^{*-1}\beta|)^{2m}$. If $\gamma = 0$, then, by Theorem 1, we have

$$c_0^{(0)}[\beta] = h^n|H|, \quad ||l_0^{(0)} \mid \widetilde{L}_2^{(m)*}(H)||^2 = |\Omega_0|\frac{h^{2m}}{(2\pi)^{2m}}\sum_{\alpha\neq 0}\frac{1}{|H^{*-1}\alpha|^{2m}}.$$

These are known results on $\widetilde{L}_2^{(m)}(H)$-optimality of "rectangle" cubature formulas and the presentation of the $\widetilde{L}_2^{(m)*}(H)$-norm of its error (see, e.g., Sobolev and Vaskevich (1996, 1997)). Let $\Omega_0 = [0, 2\pi]$ and $\omega(x) = e^{ikx}$. If k is an integer, then the explicit formula for the weights of the $\widetilde{L}_2^{(m)}[0, 2\pi]$-optimal quadrature formula was obtained in Babuška, Práger and Vitásek (1966).

The equivalent presentation for the weights of $\widetilde{L}_2^{(m)}(H)$-optimal cubature formulas was presented in Shadimetov (1999).

The Fourier coefficients of a function with one real variable are frequently calculated by the Filon quadrature formula (see, e.g., Filon (1928)). The explicit error estimates of the Filon quadrature formula are given by Petras (1990) and Köhler (1993). From Theorem 1 it follows that in case of the Sobolev spaces with periodic members the error of the $\widetilde{X}$-optimal quadrature formula is less than the error of the quadrature Filon formula.

2. Existence and Uniqueness of the $\widetilde{X}(H)$-optimal cubature formulas

Let the function $\omega(x)$ with domain R^n be locally integrable in R^n; and let

$$\omega(x + H\beta) = \omega(x), \quad \forall\beta \in Z^n.$$

[1] We will adopt this agreement on summation variables α, β in similar sums in the sequel.

We consider the error

$$(l,\varphi) \equiv \int_{\Omega_0} \omega(x)\varphi(x)\,dx - \sum_{x_k\in\Omega_0} c_k\varphi(x_k), \tag{2.1}$$

where $k = 1, \dots, N$. If $a_0 \to +\infty$, then we assume that the value of $l(x)$ on each constant functions is zero, i.e.

$$\sum_{x_k\in\Omega_0} c_k = \int_{\Omega_0} \omega(x)\,dx.$$

Hence, $l(x)$ is a member of $\widetilde{X}^*(H)$.

Theorem 2. *There exists a unique $\widetilde{X}(H)$-optimal cubature formula of the form (2.1).*

Before launching into the proof of Theorem 2, we establish the properties of the auxiliary function

$$\mathcal{E}_{\widetilde{X}}(x) = \frac{1}{|H|}\sum_{\beta}\frac{1}{A_0[\beta]}e^{i2\pi H^{*-1}\beta\cdot x}. \tag{2.2}$$

The value $A_0[\beta]$ is finite in view of (1.4). If $a_0 = +\infty$, then we agree to lead summation in (2.2) over $\beta \neq 0$. The series in (2.2) and all its derivatives converge absolutely and uniformly in $\overline{\Omega}_0$. To make sure of this, it suffices to use the inequalities

$$A_0[\beta] \geq \frac{(2\pi|H^{*-1}\beta|)^{2l}}{a_l^2}, \quad \forall l \geq 0.$$

Thus, $\mathcal{E}_{\widetilde{X}}(x)$ is an infinitely differentiable real-valued function which satisfies (1.2).

If $\Omega_0 = [0,1]$, $a_m = 1$, and $a_k \to +\infty$ for $k = 0, m+1, m+2, \dots$, then $\mathcal{E}_{\widetilde{X}}(x)$ change into the function which is proportional to the Bernoulli polynomial $B_{2m}(x)$ of degree $2m$; i.e., the "limit" equality

$$\mathcal{E}_{\widetilde{X}}(x) = \frac{(-1)^{m+1}}{(2m)!}B_{2m}(x), \quad x \in [0,1],$$

holds.

Lemma 1. *The function $\mathcal{E}_{\widetilde{X}}(x)$ is a member of $\widetilde{X}(H)$. For each error $l(x)$ of the form (2.1) the equalities*

$$(l,\varphi) = (u_l,\varphi)_{\widetilde{X}}, \quad \forall\varphi \in \widetilde{X}(H). \tag{2.3}$$

hold. Here $u_l(x)$ is a member of $\widetilde{X}(H)$ which is defined by

$$u_l(x) = \int_{\Omega_0}\omega(y)\mathcal{E}_{\widetilde{X}}(x-y)\,dy - \sum_{x_k\in\Omega_0} c_k\mathcal{E}_{\widetilde{X}}(x-x_k). \tag{2.4}$$

Proof. By the definition of $\mathcal{E}_{\widetilde{X}}(x)$, we have

$$\sum_{k=0}^{\infty} \frac{||\mathcal{E}_{\widetilde{X}}(x) \mid L_2^{(k)}(\Omega_0)||^2}{a_k^2} = \sum_{\beta} \frac{1}{A_0[\beta]} < \infty,$$

i.e., $\mathcal{E}_{\widetilde{X}}(x)$ is a member of $\widetilde{X}(H)$.

It is easy to show

$$\int\limits_{\Omega_0} \omega(y)\mathcal{E}_{\widetilde{X}}(x-y)\, dy = \frac{1}{|H|} \sum_{\beta} \frac{a_\omega[\beta]}{A_0[\beta]} e^{-i2\pi H^{*-1}\beta x},$$

with $a_\omega[\beta]$ the Fourier coefficient of $\omega(x)$. Since $\omega(x) \in L_2(\Omega_0)$, it follows that the set $\{a_\omega[\beta] \mid \beta \in Z^n\}$ is bounded. Hence, the convolution $(\mathcal{E}_{\widetilde{X}} * \omega)(x)$ is an infinitely differentiable function which satisfies (1.2). The norm square of the convolution may be written as

$$\sum_{k=0}^{\infty} \frac{||\mathcal{E}_{\widetilde{X}} * \omega(x) \mid L_2^{(k)}(\Omega_0)||^2}{a_k^2} = \frac{1}{|H|} \sum_{\beta} \frac{|a_\omega[\beta]|^2}{A_0[\beta]} < +\infty.$$

Thus, the convolution $(\mathcal{E}_{\widetilde{X}} * \omega)(x)$ and the function $u_l(x)$ which is defined by (2.4) are the members of $\widetilde{X}(H)$.

Now we consider the following series $\Phi_0(y) = \sum_{\beta} \delta(y-\beta)$, where $\delta(y)$ is the conventional Dirac delta function. It is evident that $\Phi_0(y)$ is a distribution. Moreover, $\Phi_0(y)$ is a member of $\mathcal{S}'$, the dual of the well-known Schwartz space $\mathcal{S}$. For $\forall\beta \in Z^n$ we have $\Phi_0(y+\beta) = \Phi_0(y)$. The Fourier transform of $\Phi_0(y)$ is equal to $\Phi_0(y)$, i.e., $\widetilde{\Phi}_0(\xi) = \Phi_0(y)$ (see, e.g., Sobolev (1974)). Whence and from the definition of the Fourier transform, we expand $\Phi_0(y)$ in the Fourier series as follows

$$\Phi_0(y) = \int \Phi_0(\xi) e^{i2\pi\xi y}\, d\xi = \sum_{\beta} \int \delta(\xi-\beta) e^{i2\pi\xi y}\, d\xi = \sum_{\beta} e^{i2\pi\beta y}.$$

The partial sums of this Fourier series converge to $\Phi_0(y)$ in $\mathcal{S}'$.

Let $T = (1, 1, \ldots, 1)$ be a n-tuple. Following Vladimirov (1979, p. 120), we consider the set

$$D_T' = \{f(x) \in D' \mid f(x+\beta) = f(x), \quad \forall\beta \in Z^n\}.$$

We can define the special inner product $(f, \psi)_T$ such that the generalized Parseval identity holds

$$(f,\psi)_T = \sum_{\beta} c_f^T[\beta]\overline{c_\psi^T[\beta]}, \quad \forall f \in D_T',\ \forall\psi \in C^\infty \cap D_T', \tag{2.5}$$

where $c_f^T[\beta] = (f, e^{-i2\pi\beta y})_T$ and $c_\psi^T[\beta] = (\psi, e^{-i2\pi\beta y})_T$, (see Vladimirov (1979, p. 121–124)). If $f(y) = \Phi_0(y) \in D'_T$, then

$$(\Phi_0(y), \psi(y))_T = \psi(0) \quad \text{and, therefore,} \quad c^T_{\Phi_0}[\beta] = 1 \tag{2.6}$$

(see Vladimirov (1979, p. 121)). Whence and from (2.5) it follows that

$$(\Phi_0(y), \psi(y))_T = \sum_\beta \overline{c_\psi^T[\beta]}.$$

Considering the equality

$$c_\psi^T[\beta] = (\psi, e^{-i2\pi\beta y})_T = \int_Q \psi(y) e^{-i2\pi\beta y}\, dy, \quad \psi(y) \in C^\infty \cap D'_T,$$

(see Vladimirov (1979, p. 122)), and performing the change of variable $y = H^{-1}x$, $\varphi(x) = \psi(H^{-1}x)$, we have

$$\begin{aligned}(\Phi_0(H^{-1}x), \varphi(x))_T &= \sum_\beta \overline{c_\varphi^T[\beta]} \\ &= \sum_\beta \frac{1}{|H|} \int_{\Omega_0} \varphi(x) e^{i2\pi H^{-1*}\beta x}\, dx.\end{aligned} \tag{2.7}$$

By (2.6), the equality $(\Phi_0(H^{-1}x), \varphi(x))_T = \varphi(0)$ holds. Whence and from (2.7) we infer

$$\varphi(0) = \sum_\beta \overline{c_\varphi^T[\beta]}. \tag{2.8}$$

The function $\psi(y) = \varphi(Hy)$ belongs to $C^\infty \cap D'_T$ for an arbitrary $\varphi(x) \in \widetilde{X}(H)$. Consequently, (2.8) holds for $\forall \varphi(x) \in \widetilde{X}(H)$.

From the definition of $\mathcal{E}_{\widetilde{X}}(x)$ it follows that

$$(\mathcal{E}_{\widetilde{X}}, e^{i2\pi H^{-1*}\beta x})_{\widetilde{X}} = 1, \quad \beta \neq 0. \tag{2.9}$$

The Fourier series of a function $\varphi \in \widetilde{X}(H)$ converges to φ in the norm of $\widetilde{X}(H)$. Whence and from (2.9), we have

$$(\mathcal{E}_{\widetilde{X}}, \varphi)_{\widetilde{X}} = \sum_\beta \overline{c_\varphi^T[\beta]} (\mathcal{E}_{\widetilde{X}}, e^{-i2\pi H^{-1*}\beta x})_{\widetilde{X}} = \sum_\beta \overline{c_\varphi^T[\beta]}.$$

Comparing this equality with (2.8), we see that

$$(\delta(x), \varphi(x)) = (\mathcal{E}_{\widetilde{X}}, \varphi)_{\widetilde{X}}, \quad \forall \varphi \in \widetilde{X}(H). \tag{2.10}$$

Hence, the equalities

$$(\mathcal{E}_{\widetilde{X}}(x-y),\varphi(x))_{\widetilde{X}} = (\mathcal{E}_{\widetilde{X}}(z),\varphi(z+y))_{\widetilde{X}} = \varphi(y),$$
$$(\mathcal{E}_{\widetilde{X}}(x-x_k),\varphi(x))_{\widetilde{X}} = \varphi(x_k)$$

hold; and we have

$$(u_l,\varphi)_{\widetilde{X}} = \int\limits_{\Omega_0}\omega(y)(\mathcal{E}_{\widetilde{X}}(x-y),\varphi(x))_{\widetilde{X}}\,dy - \sum_{x_k\in\Omega_0} c_k(\mathcal{E}_{\widetilde{X}}(x-x_k),\varphi(x))_{\widetilde{X}}$$
$$= \int\limits_{\Omega_0}\omega(y)\varphi(y)\,dy - \sum_{x_k\in\Omega_0} c_k\varphi(x_k) = (l,\varphi), \quad \varphi\in\widetilde{X}(H).$$

With this equality established, (2.3) is immediate. □

By the Riesz theorem every bounded linear functional $l(x)$ on the Hilbert space $\widetilde{X}(H)$ may be written as the inner product

$$(l,\varphi) = (\varphi,\psi_l)_{\widetilde{X}(H)}, \quad \varphi\in\widetilde{X}(H),$$

with $\psi_l(x)$ the uniquely determined member of $\widetilde{X}(H)$. In terms of Sobolev and Vaskevich (1996, 1997), $\psi_l(x)$ is called *the extremal function of* $l(x)$ or $\widetilde{X}(H)$-*extremal function.* Since (2.3) holds, it follows that $\psi_l(x) = u_l(x)$. Moreover, ψ_l is a solution to the following partial differential equation

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{a_k^2}\Delta^k\psi_l(x) = l(x), \tag{2.11}$$

where Δ is the Laplace operator, and the series converges in the norm of $\widetilde{X}^*(H)$. If $\widetilde{X}(H) = \widetilde{L}_2^{(m)}(H)$, then this equation is the polyharmonic equation for $\psi_l(x)$.

By (2.3), we can write the $\widetilde{X}^*(H)$-norm square of an error $l(x)$ of the form (2.1) as follows

$$||l \mid \widetilde{X}^*(H)||^2 = (l,u_l) = \int\limits_{\Omega_0}\omega(x)u_l(x)\,dx - \sum_{x_k\in\Omega_0} c_k u_l(x_k) \tag{2.12}$$
$$= \int\limits_{\Omega_0}\int\limits_{\Omega_0}\omega(x)\omega(y)\mathcal{E}_{\widetilde{X}}(x-y)\,dx dy - 2\sum_{x_k\in\Omega_0} c_k\int\limits_{\Omega_0}\omega(x)\mathcal{E}_{\widetilde{X}}(x-x_k)\,dx$$
$$+ \sum_{x_k\in\Omega_0}\sum_{x_{k'}\in\Omega_0} c_k c_{k'}\mathcal{E}_{\widetilde{X}}(x_k-x_{k'}) = \psi(c).$$

Proof of Theorem 2. Let $c = (c_1, \ldots, c_N)$ be a vector of the weights. We consider the auxiliary function

$$\psi_1(c) = \sum_{x_k \in \Omega_0} \sum_{x_{k'} \in \Omega_0} c_k c_{k'} \mathcal{E}_{\widetilde{X}}(x_k - x_{k'})$$

and check that $\psi_1(c) > 0$ for every nonzero vector $c \in R^n$.

If $c \neq 0$, then the linear combination $\delta_c(x) = \sum\limits_{k=1}^{N} c_k \delta(x - x_k)$ is a nonzero member of $\widetilde{X}^*(H)$. Hence, there exists a function $u_c(x)$ in $\widetilde{X}(H)$ such that

$$(\delta_c(x), \varphi) = (\varphi, u_c(x))_{\widetilde{X}}, \quad \forall \varphi \in \widetilde{X}.$$

Whence and from (2.10) we infer $u_c(x) = \sum\limits_{k=1}^{N} c_k \mathcal{E}_{\widetilde{X}}(x - x_k)$. Moreover, it is easy to show that

$$||\delta_c(x) \mid \widetilde{X}^*(H)||^2 = (\delta_c(x), u_c(x))$$
$$= \sum_{x_k \in \Omega_0} \sum_{x_{k'} \in \Omega_0} c_k c_{k'} \mathcal{E}_{\widetilde{X}}(x_k - x_{k'}) \equiv \psi_1(c);$$

i.e., $\psi_1(c)$ is a strictly positive function.

Since $\psi(c) \geq 0$ and $\psi_1(c) > 0$ for $c \neq 0$, it follows that $\psi(c)$ is an unbounded function in a neighborhood about infinity. Moreover, $\psi(c)$ attains a minimum in $c = c_0 \in R^n$. A rule of the form (2.1) with the weights $c = c_0$ is the $\widetilde{X}(H)$-optimal cubature formula.

We apply the strict convexity property of the unit ball in $\widetilde{X}(H)$ to prove a uniqueness of the optimal error in $\widetilde{X}^*(H)$. We recall the strict convexity property.

Let $l_1, l_2, \ldots, l_k$ be some functionals belonging to the unit sphere of $\widetilde{X}^*(H)$; and let $\mu_1, \mu_2, \ldots, \mu_k$ be nonnegative reals with sum 1. Then the respective linear combination of l_j lies in the unit ball of $\widetilde{X}^*(H)$, i.e.,

$$||\sum_{j=1}^{k} \mu_j l_j \mid \widetilde{X}^*(H)|| \leq 1.$$

Equality holds here if and only if all l_j coincide.

From strict convexity it is immediate that there is a unique optimal error in $\widetilde{X}^*(H)$. For, were it otherwise, we would find at least two errors with the same minimal norm. As follows from the last inequality, their half-sum would then have the norm less than each of them; a contradiction. $\square$

3. The difference operators of Sobolev type

In this section, we introduce the difference operators of the special form. These operators will be used to construct desired weights of $\widetilde{X}(H)$–optimal cubature formula explicitly.

Consider the auxiliary function $\Gamma_{hH}^{(\widetilde{X})}(p)$ defined by

$$\Gamma_{hH}^{(\widetilde{X})}(p)\sum_{\beta}\frac{1}{A_0[H^*p-h^{-1}\beta]}=(-1)^m,\quad p\in R^n.$$

The function $\Gamma_{hH}^{(\widetilde{X})}(p)$ is periodic in p with period matrix $h^{-1}H^{*-1}$,

$$\Gamma_{hH}^{(\widetilde{X})}(p+h^{-1}H^{*-1}\beta)=\Gamma_{hH}^{(\widetilde{X})}(p),\quad \forall\beta\in Z^n.$$

We can expand $\Gamma_{hH}^{(\widetilde{X})}(p)$ in the Fourier series

$$\Gamma_{hH}^{(\widetilde{X})}(p)=\sum_{\beta}D_{hH}^{(\widetilde{X})}[\beta]e^{-i2\pi phH\beta},$$

where

$$D_{hH}^{(\widetilde{X})}[\beta]=\frac{1}{|\Omega_1|}\int\limits_{\Omega_1}\Gamma_{hH}^{(\widetilde{X})}(p)e^{i2\pi hH\beta p}\,dp,\quad \beta\in Z^n.$$

Here the integral spreads over the fundamental parallelepiped Ω_1 of the matrix $h^{-1}H^{*-1}$.

Now we define the convolution

$$D_{hH}^{(\widetilde{X})}[\beta]*\varphi[\beta]\equiv\sum_{\gamma}D_{hH}^{(\widetilde{X})}[\beta-\gamma]\varphi[\gamma]$$

for a compactly-supported function $\varphi[\beta]$. The convolution $D_{hH}^{(\widetilde{X})}[\beta]*$ is an analog of the partial differential operator in the left side of (2.12). It is usual to use compactly-supported analogs of the linear differential operator with partial derivatives in the theory of finite differences. The convolution $D_{hH}^{(\widetilde{X})}[\beta]*$ is constructed in another fashion and not compactly-supported. If $\widetilde{X}(H)=\widetilde{L}_2^{(m)}(H)$, then the convolution $D_{hH}^{(\widetilde{X})}[\beta]*$ coincides with the operator $D_{hH}^{(m)}[\beta]*$ which was suggested by S. L. Sobolev as a difference analog for the polyharmonic operator Δ^m (see Sobolev (1974)). The properties of $D_{hH}^{(\widetilde{X})}[\beta]*$ and $D_{hH}^{(m)}[\beta]*$ are similar in many aspects. By this, we call $D_{hH}^{(\widetilde{X})}[\beta]*$ the difference operator of Sobolev type.

We have the following

Lemma 2. *$\Gamma_{hH}^{(\widetilde{X})}(p)$ is a real and analytic function for all $p \in R^n$. In a neighborhood about zero $\Gamma_{hH}^{(m)}(p)$ may be written as*

$$\Gamma_{hH}^{(\widetilde{X})}(p) = (-1)^m A_0[H^*p]\Big\{1 + \sum_{|\alpha|>0} \frac{b_\alpha}{\alpha!}(hp)^\alpha\Big\}. \tag{3.1}$$

The function $D_{hH}^{(\widetilde{X})}[\beta]$ decreases with $|\beta|$ growing. There are positive constants C and s such that for all β the inequality holds

$$|D_{hH}^{(\widetilde{X})}[\beta]| \le C|\beta|^{1/2}e^{-s|\beta|}. \tag{3.2}$$

There is a similar lemma in Sobolev and Vaskevich [12, p. 340]. For the proof in full detail, we refer to Vaskevich [13].

From Lemma 2 it follows that the convolution

$$D_{hH}^{(\widetilde{X})}[\beta] * \varphi[\beta] \equiv \sum_\gamma D_{hH}^{(\widetilde{X})}[\beta-\gamma]\varphi[\gamma]$$

exists for a function $\varphi[\beta]$ increasing with $|\beta|$ growing as degree of $|\beta|$.

From the definitions of $\Gamma_{hH}^{(\widetilde{X})}(p)$ and $D_{hH}^{(\widetilde{X})}[\beta]$ it follows that

$$\Gamma_{hH}^{(\widetilde{X})}(-p) = \Gamma_{hH}^{(\widetilde{X})}(p), \quad D_{hH}^{(\widetilde{X})}[-\beta] = D_{hH}^{(\widetilde{X})}[\beta].$$

Theorem 3. *If $1/h$ is a positive integer, then the function $\mathcal{E}_{\widetilde{X}}^{(h)}[\beta] = \mathcal{E}_{\widetilde{X}}(hH\beta)$ of a discrete variable β and the operator $D_{hH}^{(\widetilde{X})}[\beta]*$ are related by*

$$(-1)^m h^n |\Omega_0| D_{hH}^{(\widetilde{X})}[\beta] * \mathcal{E}_{\widetilde{X}}^{(h)}[\beta] = \Phi_h[\beta]. \tag{3.9}$$

Here $\Phi_h[\beta]$ equals 1 in the points of set $\{\alpha/h \mid \alpha \in Z^n\}$ and 0 outside of the set. If $\delta[\beta] = 0$ for $\beta \ne 0$ and $\delta[0] = 1$, then

$$\Phi_h[\beta] = \sum_\gamma \delta[\beta - \gamma/h].$$

Proof. By the definition of $\mathcal{E}_{\widetilde{X}}(x)$, we have

$$\sum_\alpha \mathcal{E}_{\widetilde{X}}^{(h)}[\alpha]e^{-i2\pi hH^*p\alpha} = \frac{1}{|H|}\sum_\beta \frac{1}{A_0[\beta]}\sum_\alpha e^{-i2\pi h(H^*p+\beta)\alpha}. \tag{3.10}$$

Substituting the equality

$$\sum_\alpha e^{-i2\pi h(H^*p+\beta)\alpha} = \Phi_0(hH^*p + h\beta) = \sum_\alpha \delta(hH^*p + h\beta - \alpha)$$

into (3.10), we interchange the order of summation over β and α. So, the sum in the left side of (3.10) equals

$$\frac{1}{|H|}\sum_{\alpha}\frac{1}{A_0[H^*p-h^{-1}\alpha]}\sum_{\beta}\delta(hH^*p+h\beta-\alpha).$$

Given a positive integer $1/h$, it is easy to show

$$\sum_{\beta}\delta(hH^*p+h\beta-\alpha)=\sum_{\beta'}\delta(hH^*p-h\beta').$$

Consequently, the product $\Gamma_{hH}^{(\widetilde{X})}(p)\sum\limits_{\alpha}\mathcal{E}_{\widetilde{X}}^{(h)}[\alpha]e^{-i2\pi hH^*p\alpha}$ equals

$$\frac{1}{|H|}\Gamma_{hH}^{(\widetilde{X})}(p)\Big(\sum_{\alpha}\frac{1}{A_0[H^*p-h^{-1}\alpha]}\Big)\Big(\sum_{\beta}\delta(hH^*p-h\beta)\Big). \tag{3.11}$$

By the definition of $\Gamma_{hH}^{(\widetilde{X})}(p)$, we have

$$\Gamma_{hH}^{(\widetilde{X})}(p)\sum_{\beta}\frac{1}{A_0[H^*p-h^{-1}\beta]}=(-1)^m,\quad p\in R^n. \tag{3.12}$$

Whence and from (3.11), we infer

$$\Gamma_{hH}^{(\widetilde{X})}(p)\sum_{\alpha}\mathcal{E}_{\widetilde{X}}^{(h)}[\alpha]e^{-i2\pi hH^*p\alpha} \tag{3.13}$$

$$=\frac{(-1)^m}{|H|}\sum_{\beta}\delta(hH^*p-h\beta)=\frac{(-1)^m}{h^n|H|}\sum_{\beta}e^{-i2\pi phH(\beta/h)}.$$

The Fourier coefficients of the product of two functions result from convoluting the Fourier coefficients of the factors. Considering this and substituting the expansion

$$\Gamma_{hH}^{(\widetilde{X})}(p)=\sum_{\beta}D_{hH}^{(\widetilde{X})}[\beta]e^{-i2\pi phH\beta}$$

in (3.13), we come to (3.9). $\square$

4. The weights of the $\widetilde{X}(H)$-optimal cubature formulas

In this section, we consider the cubature formulas of the form

$$(l,\varphi)=\int\limits_{\Omega_0}\omega(x)\varphi(x)\,dx-\sum_{hH\beta\in\Omega_0}c[\beta]\varphi(hH\beta), \tag{4.1}$$

where $\omega(x)$ is the weight function that was introduced in section 2.

Theorem 4. *The $\widetilde{X}(H)$-optimal cubature formula of the form (4.1) has the following weights*

$$c_0^{(\omega)}[\beta]=h^n(-1)^m|H|\int\limits_{\Omega_0}\omega(x)D_{hH}^{(\widetilde{X})}[\beta]*\mathcal{E}_{\widetilde{X}}(x-hH\beta)\,dx, \tag{4.2}$$

where $hH\beta$ is a vector in Ω_0.

Proof. Let $c=\{c[\beta]\mid hH\beta\in\Omega_0\}$ be the vector such that the quadratic form $\psi(c)$ defined by (2.12) attains a minimum in c. The vector c is a unique solution to

$$\sum_{hH\beta'\in\Omega_0}c[\beta']\mathcal{E}_{\widetilde{X}}(hH\beta-hH\beta')=f[\beta],\quad hH\beta\in\Omega_0, \tag{4.3}$$

where

$$f[\beta]=\int\limits_{\Omega_0}\omega(x)\mathcal{E}_{\widetilde{X}}(x-hH\beta)\,dx.$$

If $c[\beta]=0$ for $hH\beta\notin\overline{\Omega}_0$, then we can change the sum over β' in (4.3) by the sum over $\beta'\in Z^n$. Considering this, we write (4.3) can be written in the convolution form

$$\mathcal{E}_{\widetilde{X}}^{(h)}[\beta]*c[\beta]=f[\beta],\quad h\beta\in Q, \tag{4.4}$$

where $\mathcal{E}_{\widetilde{X}}^{(h)}[\beta]\equiv\mathcal{E}_{\widetilde{X}}(hH\beta)$. Let $v[\beta]=\mathcal{E}_{\widetilde{X}}^{(h)}[\beta]*c[\beta]$. By the periodicity of $\mathcal{E}_{\widetilde{X}}(x)$, we infer $v[\beta+h^{-1}\delta]=v[\beta]$ and $f[\beta+h^{-1}\delta]=f[\beta]$ for $\forall\delta\in Z^n$. Hence, $v[\beta]$ and $f[\beta]$ are completely defined by the values $\{v[\beta]\mid h\beta\in Q\}$ and $\{f[\beta]\mid h\beta\in Q\}$ respectively. Whence and from (4.4) it follows that

$$\mathcal{E}_{\widetilde{X}}^{(h)}[\beta]*c[\beta]=f[\beta],\quad\forall\beta\in Z^n. \tag{4.5}$$

Now we apply the convolution operator $D_{hH}^{(\widetilde{X})}[\beta]*$ to both the sides of (4.5) and obtain

$$(D_{hH}^{(\widetilde{X})}[\beta]*\mathcal{E}_{\widetilde{X}}^{(h)}[\beta])*c[\beta]=D_{hH}^{(\widetilde{X})}[\beta]*f[\beta],\quad\forall\beta\in Z^n.$$

Inserting (3.9) into the left part of this equality, we infer

$$c_0^{(\omega)}[\beta] \equiv c[\beta] = \sum_\alpha c[\beta - h^{-1}\alpha]$$

$$= \Phi_h[\beta] * c[\beta] = (-1)^m |\Omega_0| h^n D_{hH}^{(\widetilde{X})}[\beta] * f[\beta].$$

With this equality established, (4.2) is immediate. □

Corollary 1. *The weights $c_0^{(\omega)}[\beta]$ of the $\widetilde{X}(H)$-optimal cubature formula of the form (4.1) may be written as follows*

$$c_0^{(\omega)}[\beta] = h^n \sum_\alpha \frac{(-1)^m \Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\alpha)}{A_0[\alpha]} e^{-i2\pi h\beta\alpha} a_\omega[\alpha], \tag{4.6}$$

with $hH\beta$ a vector in Ω_0 and $a_\omega[\alpha]$ the Fourier coefficient of the weight function $\omega(x)$.

Proof. The weight function $\omega(x)$ which was introduced in section 2 is a locally integrable function. Whence and from the periodicity of $\Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\delta)$, it follows that the series in (4.6) converges absolutely and uniformly. Applying to (2.2) the convolution operator $D_{hH}^{(\widetilde{X})}[\beta]*$, we have

$$\begin{aligned} &D_{hH}^{(\widetilde{X})}[\beta] * \mathcal{E}_{\widetilde{X}}(x - hH\beta) \\ &= \frac{1}{|H|} \sum_\alpha \frac{1}{A_0[\alpha]} D_{hH}^{(\widetilde{X})}[\beta] * e^{-i2\pi H^{*-1}\alpha(x-hH\beta)}. \end{aligned} \tag{4.7}$$

By the definitions of $D_{hH}^{(\widetilde{X})}[\beta]*$ and $\Gamma_{hH}^{(\widetilde{X})}(p)$, we infer

$$D_{hH}^{(\widetilde{X})}[\beta] * e^{-i2\pi H^{*-1}\alpha(x-hH\beta)}$$

$$= e^{-i2\pi H^{*-1}\alpha(x-hH\beta)} \sum_\sigma D_{hH}^{(\widetilde{X})}[\sigma] e^{-i2\pi H^{*-1}\alpha\cdot hH\sigma}$$

$$= e^{-i2\pi H^{*-1}\alpha(x-hH\beta)} \Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\alpha).$$

Whence and from (4.7) it follows that the convolution $D_{hH}^{(\widetilde{X})}[\beta] * \mathcal{E}_{\widetilde{X}}(x - hH\beta)$ equals

$$\frac{1}{|H|} \sum_\alpha \frac{\Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\alpha)}{A_0[\alpha]} e^{-i2\pi H^{*-1}\alpha(x-hH\beta)}. \tag{4.8}$$

Substituting (4.8) into (4.2), we interchange the integration over x and the summation over α. By choosing $-\alpha$ as a new variable of the summation, we infer (4.6). □

5. The error of the $\widetilde{X}(H)$-optimal cubature formulas

Proof of Theorem 1. Let $\omega(x) = e^{i2\pi H^{-1*}\gamma x}$. By (4.6), we can write the weights of the $\widetilde{X}(H)$-optimal cubature formula of the form (4.1) as follows

$$c_0^{(\gamma)}[\beta] = h^n|H|\frac{(-1)^m\Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\gamma)}{A_0[\gamma]}e^{i2\pi h\beta\gamma}.$$

With this equality established, (1.5) is immediate.

We estimate the error of the cubature formula with the weights $c_0^{(\gamma)}[\beta]$ on every space of the family $\{\widetilde{X}^*(H) \mid (a_k)\}$.

Given an error of the form (4.1), from (2.12) we have

$$\begin{aligned}(5.1)\qquad ||l \mid \widetilde{X}^*||^2 &= (l, u_l)\\ &= ||u_l \mid \widetilde{X}||^2 = |H|\sum_\alpha A_0[\alpha]|a_{u_l}[\alpha]|^2.\end{aligned}$$

The Fourier coefficient $a_{u_l}[\alpha]$ of the extremal function $u_l(x)$ depends on $L[\alpha] \equiv (l(x), e^{i2\pi H^{-1*}\alpha x})$. By (2.3) with $\varphi(x) = e^{i2\pi H^{-1*}\alpha x}$, we have

$$\begin{aligned}L[\alpha] &= (u_l(x), e^{-i2\pi H^{*-1}\alpha x})_{\widetilde{X}}\\ &= \sum_{k=0}^{\infty}\frac{|2\pi H^{*-1}\alpha|^{2k}}{a_k^2}\int_{\Omega_0} u_l(x)e^{i2\pi H^{*-1}\alpha x}\,dx = \frac{1}{|H|}A_0[\alpha]a_{u_l}[\alpha].\end{aligned}$$

Apply first this equality and next (5.1). Then we infer

$$(5.2)\qquad ||l \mid \widetilde{X}^*||^2 = \frac{1}{|H|}\sum_\alpha \frac{|L[\alpha]|^2}{A_0[\alpha]}.$$

Based on (5.2), we find the $\widetilde{X}^*(H)$-norm of the $\widetilde{X}(H)$-optimal error explicitly. Let us compute the Fourier coefficients $L[\alpha]$, $\alpha \in Z^n$, of the optimal error.

Given the weights $c_0^{(\omega)}[\beta]$, we denote by $L_0^{(\omega)}[\alpha]$ the respective Fourier coefficients $L[\alpha]$. If $\omega(x) = e^{i2\pi H^{-1*}\gamma x}$, then we agree to write $L_0^{(\gamma)}[\alpha]$ instead of $L_0^{(\omega)}[\alpha]$. Apply first the definition of $L_0^{(\omega)}[\alpha]$ and next (4.2). Then, by (5.1), we infer

$$\begin{aligned}L_0^{(\omega)}[\alpha] &= a_\omega[-\alpha] - \sum_{hH\beta\in\Omega_0} c_0^{(\omega)}[\beta]e^{-i2\pi H^{*-1}\alpha\cdot hH\beta} = a_\omega[-\alpha]\\ &- \int_{\Omega_0}\omega(x)\Big[h^n\sum_{h\beta\in Q} e^{i2\pi h\beta(\sigma-\alpha)}\Big]\sum_\sigma \frac{(-1)^m\Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\sigma)}{A_0[\sigma]}e^{-i2\pi H^{*-1}\sigma x}\,dx.\end{aligned}$$

Let $(\sigma - \alpha)h$ be a vector with the integer entries. Then

$$\Big[h^n \sum_{h\beta \in Q} e^{i2\pi h\beta\cdot(\sigma-\alpha)}\Big] = 1.$$

Otherwise, the value in brackets equals zero. Hence

$$L_0^{(\omega)}[\alpha] = a_\omega[-\alpha] - \sum_\sigma \frac{(-1)^m \Gamma_{hH}^{(\widetilde{X})}(H^{*-1}(\alpha+\sigma/h))}{A_0[\alpha+\sigma/h]} a_\omega[-\alpha-\sigma/h].$$

Let $\omega(x) = e^{i2\pi H^{-1*}\gamma\cdot x}$. Then

$$L_0^{(\gamma)}[\gamma] = |H|\Big[1 - \frac{(-1)^m \Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\gamma)}{A_0[\gamma]}\Big], \tag{5.3}$$

$$L_0^{(\gamma)}[\gamma+\sigma/h] = -|H|\frac{(-1)^m \Gamma_{hH}^{(\widetilde{X})}(H^{*-1}\gamma)}{A_0[\gamma]}, \quad \sigma \neq 0, \sigma \in Z^n.$$

If there exists a noninteger entry of a vector $h(\gamma-\alpha)$, then $L_0^{(\gamma)}[\alpha]$ equals zero.

Given an $\widetilde{X}$-optimal cubature formula of the form (4.1), we intend to find the $\widetilde{X_1}^*$-norm of the $\widetilde{X}$-optimal error $l_0^{(\gamma)}(x)$. Here $\widetilde{X_1}$ is a Hilbert space from the family under consideration. Let $\widetilde{A}_0[\beta]$ be the function (1.4) in case of the space $\widetilde{X_1}$. From (5.2) and (5.3) it follows that $||l_0^{(\gamma)} \mid \widetilde{X_1}^*||^2$ equals the product of

$$|H|\left[\frac{A_0^2[\gamma]}{\widetilde{A}_0[\gamma]}\Big(\sum_{\beta\neq 0}\frac{1}{A_0[\gamma-\beta/h]}\Big)^2 + \sum_{\beta\neq 0}\frac{1}{\widetilde{A}_0[\gamma-\beta/h]}\right]$$

and

$$\left[1 + \sum_{\beta\neq 0}\frac{A_0[\gamma]}{A_0[\gamma-\beta/h]}\right]^{-2}.$$

If $\widetilde{X_1} = \widetilde{X}$, then we have (1.6). □

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