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Optimal systems of symmetry subalgebras for big models of gasdynamics

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Summary. For an arbitrary state equation the gasdynamics equations admit the 11-dimensional Lie algebra of transformations. But there are 13 special state equations for which the corresponding Lie algebra of transformations may be expanded. In this event we call the gasdynamics equations *the big models*. Some of the Lie algebras under consideration are similar to one another and the algebra structure may be different under further specialization of the state equation. It was noted that only two specializations of the state equation need to be added into consideration. In the paper we make a review of extensions for which the optimal systems of subalgebras may be found and calculate these optimal systems. Each of the extensions is presented as the semidirect sum of the Abelian 6-dimensional ideal and a subalgebra. The normalized optimal systems of these subalgebras are calculated.

Key words: gasdynamics, state equation, Lie algebra of transformations, optimal system of subalgebras

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1. Introduction

The gasdynamics equations are as follows

$$(1) \qquad \rho D\mathbf{u} + \nabla p = 0, \quad D\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \quad Dp + \rho c^2 \mathrm{div}\,\mathbf{u} = 0,$$

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where $\mathbf{u}$, ρ, and p denote the velocity, density, and pressure; $D = \partial_t + \mathbf{u}\cdot\nabla$; $c^2(\rho, p) = f_\rho$ is the square of the sound velocity; $p = f(\rho, S)$ is the state equation, and S is the entropy. For an arbitrary state equation the gasdynamics equations admit the 11-dimensional Lie algebra of operators L_{11} [1]. In the Cartesian coordinate system, the operators

$$
(2)\quad \begin{aligned}
X_1 &= \partial_x,\ X_2 = \partial_y,\ X_3 = \partial_z,\ X_4 = t\partial_x + \partial_u,\ X_5 = t\partial_y + \partial_v,\\
X_6 &= t\partial_z + \partial_w,\ X_7 = y\partial_z - z\partial_y + v\partial_w - w\partial_v,\\
X_8 &= z\partial_x - x\partial_z + w\partial_u - u\partial_w,\ X_9 = x\partial_y - y\partial_x + u\partial_v - v\partial_u,\\
X_{10} &= \partial_t,\ X_{11} = t\partial_t + x\partial_x + y\partial_y + z\partial_z
\end{aligned}
$$

constitutes the basis of L_{11}. For the special state equations the admissible algebra may be expanded into the semidirect sum of nontrivial summands and the exhausting list of these expansions is given in Table 1. Here E is an arbitrary function of the entropy, N is the number

Table 1.

N	p	k	Complementary operators to L_{11}
1	$f(\rho, S)$	11	—
2	$\rho^\gamma f(E\rho)$, $\gamma \neq 0, 1$	12	$Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p$, $\gamma_1 = 2\gamma(\gamma-1)^{-1}$, $\gamma_1 \neq 0, 2$
3	$\rho f(E\rho)$	12	Y_p
4	$f(E\rho)$	12	$Y_0 + 2\rho\partial_\rho$
5	$Ef(\rho)$	12	$Y_0 - 2p\partial_p$
6	$E\rho^\gamma$	13	$Y_0 + 2\rho\partial_\rho$, Y_p
7	$E\rho^{5/3}$	14	Y_p, $Y_0 + 2\rho\partial_\rho$, $xX_4 + yX_5 + zX_6 + t(Y_0 - 3\rho\partial_\rho - 5p\partial_p)$
8	$\ln\rho + f(E\rho)$	12	$Y_0 + 2\rho\partial_\rho + 2\partial_p$
9	$f(\rho) + E$	12	Y_1
10	$\rho^\gamma + E$, $\gamma \neq 0, 1$	13	Y_1, $Y_0 - (\gamma_1 - 2)\rho\partial_\rho - \gamma_1 p\partial_p$
11	$\rho + E$	13	Y_1, Y_p
12	$\ln\rho + E$	13	Y_1, $Y_0 + 2\rho\partial_\rho$
13	E	∞	$Y_0 + 2\rho\partial_\rho$, $Y_{\varphi(p)} = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p$

of the admitting algebra; $Y_0 = t\partial_t - u\partial_u - v\partial_v - w\partial_v - w\partial_w$; φ and f are arbitrary functions; γ is a parameter; k is the dimension of the expanded Lie algebra. The terms containing variables ρ and p in the expanding operators are marked out so that it is convenient to determine the similar algebras.

2. Similar algebras

Two algebras of operators are similar to one another if the basis operators of the first algebra transform into the basis operators of the second algebra by a change of variables p' and ρ'. The structure of similar algebras is the same since they have the same table of commutators. The algebra with $N = 6$ (resp. $N = 3$) is similar to the algebra with $N = 12$ (resp. $N = 9$); and the algebra with $N = 2$ is similar to the algebras with $N = 4$, $N = 5$, and $N = 8$. The corresponding changes of variables are given in Table 2. It is necessary

Table 2.

Algebras	Similarity transformations
$N = 3,\ N = 9$	$\rho' = p\rho^{-1},\ p' = \ln p$
$N = 4,\ N = 5$	$\rho' = p,\ p' = \rho^{-1}$
$N = 4,\ N = 8$	$\rho = \rho',\ p' = p + \ln\rho$
$N = 2,\ N = 5$	$\rho' = p\rho^{-\gamma},\ p' = p\rho^{-1}$
$N = 6,\ N = 12$	$\rho' = \rho p^{-1},\ p' = \ln p$

to construct the optimal systems of subalgebras for the basis algebras from Table 1 in order to list the different gasdynamics submodels. By a definition, the optimal system consists of all subalgebras which are not conjugated with respect to the inner automorphisms; and the normalizers of these subalgebras must be also in the optimal system.

Let H be the subalgebra of the Lie algebra L. By a definition, the normalizer $N_L(H)$ of H is the maximal subalgebra of L for which H is an ideal. Hence the submodel according with H also admits the factor algebra $N_L(H)/H$.

It follows from Table 2 that it is sufficient to construct the optimal systems for the algebras with $N = 1$, 9, 4, 11, 12, 13, 7, and 10.

For $N = 13$ the infinite algebra decomposes into the direct sum of the ideals $L_\infty = \{Y_{\varphi(p)}\}$ and $J = L_{11}\dot{\oplus}\{Y_0 + 2\rho\partial_\rho\}$, where J is the semidirect sum of the ideal L_{11} and the 1-dimensional subalgebra $\{Y_0 + 2\rho\partial_\rho\}$. The ideal J is enumerated by $N = 4$ in Table 1.

Theorem 1. *The optimal system of the finite subalgebras of the infinite algebra L_∞ consists of*

$$\{Y_1\}, \quad \{Y_1, Y_p\}, \quad \{Y_1, Y_p, Y_{p^2}\}, \tag{3}$$

where $\{Y_1\}$, $\{Y_1, Y_p\}$, and $\{Y_1, Y_p, Y_{p^2}\}$ are nontrivial finite subalgebras with respect to the similarity transformation.

Proof. Let $c = 0$ and $\psi'(p) \neq 0$. Then (1) admits the similarity transformations $\rho' = \rho\psi'(p)$ and $p' = \psi(p)$ of L_∞. These transformations do not change the operators from L_{11} and $Y_0 + 2\rho\partial_\rho$. For the operator $Y_{\varphi(p)}$ of L_∞ the equalities hold

$$Y_{\varphi(p)} = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p = Y'_{\chi(p')} = \rho'\chi'(p')\partial_{\rho'} + \chi(p')\partial_{p'},$$

where $\chi(p') = \varphi(p)\psi'(p)$. Using the change of variables

$$\psi(p) = \int \varphi^{-1}(p)\, dp$$

it is easy to show that $Y_{\varphi(p)}$ is similar to Y_1'. Hence each 1-dimensional subalgebra is similar to the translation.

In an arbitrary 2-dimensional subalgebra there is a basis operator such that the similarity transformation moves it into the translation Y_1; and for another basis operator $Y_{\varphi(p)}$ the following subalgebra equation holds

$$[Y_1, Y_\varphi] = aY_1 + bY_\varphi,$$

with a and b constants. Hence $\varphi' = -a - b\varphi$; and for $b = 0$ we have $\varphi = p$ within a basis interchange. If $b \neq 0$ then we can use the change of bases to obtain $\varphi = e^{-p}$. Using the similarity transformation with the function $\psi = \ln p$ it is easy to show that each 2-dimensional subalgebra is similar to $\{Y_1, Y_p\}$.

In an arbitrary 3-dimensional subalgebra there is a basis operator such that the similarity transformation moves it into the translation Y_1; and for two other basis operators Y_{φ_1} and Y_{φ_2} the following subalgebra equations hold

$$\begin{aligned}
[Y_1, Y_{\varphi_1}] &= b_1Y_1 + a_{11}Y_{\varphi_1} + a_{12}Y_{\varphi_2},\\
[Y_1, Y_{\varphi_2}] &= b_2Y_1 + a_{21}Y_{\varphi_1} + a_{22}Y_{\varphi_2},\\
[Y_{\varphi_1}, Y_{\varphi_2}] &= b_3Y_1 + c_1Y_{\varphi_1} + c_2Y_{\varphi_2},
\end{aligned}$$

where b_i, a_{ij}, and c_i are constants. We assume that (a_{ij}) has the Jordan form; otherwise, we can pass to the suitable basis of the algebra. In this event the subalgebra equations have the following form

$$\begin{gathered}
\varphi_1' = b_1 + a_{11}\varphi_1 + a_{12}\varphi_2, \quad \varphi_2' = b_2 + a_{21}\varphi_1 + a_{22}\varphi_2,\\
b_3 + (c_1 - b_2)\varphi_1 + (c_2 + b_1)\varphi_2\\
= a_{21}\varphi_1^2 + (a_{11} + a_{22})\varphi_1\varphi_2 - a_{12}\varphi_2^2.
\end{gathered} \tag{4}$$

Let $(a_{ij}) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Then $a \neq 0$, $b \neq 0$, and $a \neq b$. For, were it otherwise, the basis operators would be linearly dependent. It is not hard to establish on using the suitable basis of the algebra that $b_1 =$

$b_2 = 0$. Whence and from (4) it follows that $\varphi_1 = e^{ap}$, $\varphi_2 = e^{bp}$, $b_3 + c_1 e^{ap} + c_2 e^{bp} = (b-a)e^{(a+b)p}$. The last equality yields $a+b=0$, $b_3 = b-a$, $c_1 = c_2 = 0$. Thus we arrive at the subalgebra $\{Y_{e^{-ap}}, Y_1, Y_{e^{ap}}\}$, which is transformed to the 3-dimensional subalgebra of the form (3) by means of the similarity transformation with $\psi(p) = e^{ap}$.

Let $(a_{ij}) = \begin{pmatrix} a & 0 \\ 2 & a \end{pmatrix}$; and $a \neq 0$. Using the suitable basis of the algebra we derive $b_1 = b_2 = 0$. Whence and from (4) it follows that $\varphi_1 = e^{ap}$, $\varphi_2 = 2pe^{ap}$, and $b_3 + (c_1 + 2pc_2)e^{ap} = 2(1+2ap)e^{2ap}$. The last equality is inconsistent.

Let $a = 0$. Using the suitable basis of the algebra we derive $b_2 = 0$ and $b_1 = 1$. Whence and from (4) it follows that $\varphi_1 = p$, $\varphi_2 = p^2$, $c_1 = 1$, and $c_1 = b_3 = 0$. Thus we arrive at the subalgebra of the form (3).

Let $(a_{ij}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $b \neq 0$. Using the suitable basis of the algebra we derive $b_1 = b_2 = 0$. Whence and from (4) it follows that

$$\varphi_1 = e^{ap}\cos(bp), \quad \varphi_2 = -e^{ap}\sin(bp),$$

and $b_3 e^{-ap} + c_1\cos(bp) - c_2\sin(bp) + b + a\sin(2ap) = 0$. Hence $b = 0$; a contradiction.

Given a finite subalgebra of L_∞ we now prove that its dimension is not greater than 3. On the contrary; let the basis of the subalgebra be Y_1, Y_{φ_i}, $i = 1, \ldots, k$, where 1, φ_i are linear independent functions, and $k \geq 3$. Since the following subalgebra equations hold

$$[Y_1, Y_{\varphi_i}] = b_i Y_1 + a_{ij} Y_{\varphi_j}, \quad [Y_{\varphi_i}, Y_{\varphi_j}] = b_{ij} Y_1 + c_{ijk} Y_{\varphi_k}, \; i \neq j,$$

it follows that

$$(5) \quad \varphi_i' = b_i + a_{ij}\varphi_{\varphi_j}, \; \varphi_i(b_j + a_{jl}\varphi_l) - \varphi_j(b_i + a_{il}\varphi_l) = b_{ij} + c_{ijl}\varphi_l.$$

We assume that (a_{ij}) is equal to $\begin{pmatrix} a & & 0 \\ 1 & \ddots & \\ & \ddots & \ddots \\ 0 & & 1 \; a \end{pmatrix}$, i. e., (a_{ij}) has the Jordan form. Otherwise, we pass to the suitable basis of the algebra. By the same reason, for $a \neq 0$ we can assume that $b_i = 0$, $i = 1, \ldots, k$. In this event it follows from (5) that $\varphi_l = l!p^l e^{ap}$, $l = 0, \ldots, k-1$; $b_{ij} = 0$, $c_{ijl} = 0$, and $i!(j-1)! - j!(i-1)! = 0$. Thus we again arrive at a contradiction.

Let $a = 0$. Using the suitable basis of the algebra we derive $b_1 = 1$ and $b_i = 0$ for $i > 1$. Whence and from (5) it follows that $\varphi_l = (l!)p^l$, $l = 1, \ldots, k$; $b_{ij} = 0$, $c_{ijl} = 0$; $i = j$; a contradiction.

Thus (a_{ij}) must be a diagonal matrix, i. e., $(a_{ij}) = \mathrm{diag}(a_1, \dots, a_k)$, where $a_i \neq 0$ and $a_i = a_j$ for $i \neq j$. Moreover, we can assume that $b_i = 0$. From (5) it follows that

$$\varphi_i = e^{a_i p}, \quad (a_j - a_i)e^{(a_i+a_j)p} = b_{ij} + c_{ijl}e^{a_l p}.$$

Let us assume that there exists a pair (i,j) such that $a_i + a_j = 0$. Let this pair be $(1,2)$. Then for $(i,j) \neq (1,2)$ there exists a number l such that $1 \leq l \leq k$ and $a_l = a_i + a_j$. From these equalities it follows that $la_1 + a_3 = a_2$ for $l > 0$, $ma_2 + a_3 = a_1$ for $m > 0$, and $a_3 = (m+1)a_1 = -(l+1)a_1$. Hence $a_1 = 0$; a contradiction.

Hence there does not exist any pair (i,j) such that $a_i + a_j = 0$. Then $la_1 + a_2 = na_1 + a_2$ for $l > n$, which is again a contradiction.

We assign the number $N = 12$ to the algebra $J \oplus \{Y_1\}$, the number $N = 14$ to the algebra $J \oplus \{Y_1, Y_p\}$ and the number $N = 15$ to the algebra $J \oplus \{Y_1, Y_p, Y_{p^2}\}$.

The optimal systems for algebras with $N = 1$, 7, 12, and $N = 10$ are constructed in [1–4]. If the structure constants of the algebra are the functions of $\gamma_1 = 2\gamma(\gamma-1)^{-1}$, $\gamma_1 \neq 0$, 2, then we study the algebra with $N = 10$ and extend the corresponding optimal systems for $\gamma_1 = \pm 1$ in [4].

In Section 3 we construct the optimal systems of the finite subalgebra for algebras with $N = 9$,11;4,14, and $N = 15$.

3. Inner automorphisms

Given the Lie algebra, we construct the optimal system of its subalgebras with the help of the inner automorphism group and the inversion group corresponding to inversions admitted by (1). The transformations are found with the aid of the commutator table.

The algebra L_{11} is an ideal of algebras with $N = 9$, 11; 4, 14, and $N = 15$. Therefore the table of commutators for L_{11} is separately written in Table 3. Instead of the operators X_i, $i = 1, \dots, 11$; $X_{12} = Y_0 + 2\rho\partial_\rho$, Y_1, Y_p, Y_{p^2}, it is convenient to write the subscripts i, 12, $\bar{1}$, p, and p^2. The empty space in Table 3 means that the corresponding commutator is equal to 0. It is convenient to introduce the decompositions of the algebras under consideration in the direct sum of the ideals and semidirect sum of the ideal and the subalgebra

$$N = 9\colon\ L_{11} \oplus \{\bar{1}\} = J_6 \dot{\oplus} (J_3 \oplus \{10, 11\} \oplus \{\bar{1}\}),$$

where $\{\bar{1}\}$ is the center, $J_6 = \{1,2,3,4,5,6\}$ is the Abelian ideal, $J_3 = \{7,8,9\}$ is the simple subalgebra of rotations;

$$N = 11\colon\ L_{11} \oplus \{1, \bar{p}\} = J_6 \dot{\oplus} (J_3 \oplus \{10, 11\} \oplus \{\bar{1}, p\});$$

Table 3.

	1	2	3	4	5	6	7	8	9	10	11
1								-3	2		1
2							3		-1		2
3							-2	1			3
4								-6	5	-1	
5							6		-4	-2	
6							-5	4		-3	
7		-3	2		-6	5		-9	8		
8	3		-1	6		-4	9		-7		
9	-2	1		-5	4		-8	7			
10				1	2	3					10
11	-1	-2	-3							-10	

$$N = 4\colon\ L_{11}\dot{\oplus}\{12\} = J_6\dot{\oplus}(J_3 \oplus (\{10\}\dot{\oplus}\{11,12\}));$$

$$N = 14\colon\ (L_{11}\dot{\oplus}\{12\}) \oplus \{\bar{1}, p\} = J_6\dot{\oplus}(J_3 \oplus (\{10\}\dot{\oplus}\{11,12\}) \oplus \{\bar{1}, p\});$$

$$N{=}15\colon (L_{11}\dot{\oplus}\{12\})\oplus\{\bar{1}, p, p^2\} = J_6\dot{\oplus}(J_3\oplus(\{10\}\dot{\oplus}\{11,12\})\oplus\{\bar{1}, p, p^2\}).$$

The commutators of the extensions of L_{11} are given in Table 4. Let L be an algebra. For each operator $Y \in L$ we can calculate the

Table 4.

	1	2	3	4	5	6	7	8	9	10	11	12	$\bar{1}$	p	p^2
12				4	5	6				-10					
$\bar{1}$														$\bar{1}$	2p
p													$-\bar{1}$		p^2
p^2													$-2p$	$-p^2$	

one-parameter group of the inner automorphisms of L. To this end we must solve the initial value problem

$$\partial_a X' = [X', Y],\ \ X'\big|_{a=0} = X \tag{6}$$

for each basis operator Y and a corresponding to Y. Let $X = \sum\limits_{i=1}^{12} x^i X_i + y^0 Y_{\bar{1}} + y^1 Y_p + y^2 Y_{p^2}$. Then the transformations of the one-parameter group for algebras with $N=4$, 14, and $N=15$ are defined as follows

$$
(7)\qquad
\begin{array}{lll}
T & : p_1(x') = p_1(x) + x^{11}\boldsymbol{\alpha}_6 - \boldsymbol{\alpha}_1 \times p_3(x); & \\
\mathit{\Gamma} & : p_1(x') = p_1(x) - x^{10}\boldsymbol{\alpha}_2; & \\
 & \quad p_2(x') = p_2(x) - \boldsymbol{\alpha}_4 \times p_3(x) - x^{12}\boldsymbol{\alpha}_2; & \\
S & : p_1(x') = Sp_1(x), \quad p_2(x') = Sp_2(x), & \\
 & \quad p_3(x') = Sp_3(x); & \\
A_{10} & : p_1(x') = p_1(x) + a_{10}p_2(x), & \\
 & \quad x'^{10} \quad = x^{10} + a_{10}(x^{11} + x^{12}); & \\
A_{11} & : p_1(x') = a_{11}p_1(x), \quad x'^{10} = a_{11}x^{10}; & \\
A_{12} & : p_2(x') = a_{12}p_2(x), \quad x'^{10} = a_{12}^{-1}x^{10}; & \\
B_0 & : y'^0 \quad = y^0 - y^1 b_0 + y^2 b_0^2, \quad y'^1 = y^1 - 2y^2 b_0; & \\
B_1 & : y'^0 \quad = y^0 b_1, \quad y'^2 = y^2 b_1^{-1}; & \\
B_2 & : y'^1 \quad = y^1 + 2y^0 b_2, \;\; y'^2 = y^2 + y^1 b_2 + y^0 b_2^2; & \\
\epsilon_1 & : p_1(x') = -p_1(x), \quad p_2(x') = -p_2(x); & \\
\epsilon_2 & : p_2(x') = -p_2(x), \quad x'^{10} = -x^{10}. &
\end{array}
$$

Here $p_1(x) = (x^1, x^2, x^3)$, $p_2(x) = (x^4, x^5, x^6)$, $p_3(x) = (x^7, x^8, x^9)$; $\boldsymbol{\alpha}_1 = (a_1, a_2, a_3)$, $\boldsymbol{\alpha}_2 = (a_4, a_5, a_6)$; S is an arbitrary rotation in R^3 depending on parameters a_7, a_8, a_9; b_0, b_1, b_2 are parameters of the groups corresponding to the basis operators Y_1, Y_p, Y_{p^2}; ϵ_1, ϵ_2 are inversions admitted by Table 3; $\times$ is the symbol of the vector product in R^3.

If we cross out A_{12} and B_2 from (7) and assume that x^{12} and y^2 are vanishing then we obtain the inner automorphisms for the algebras with $N=9$ and $N=11$.

Remark 1. There is an invariant $(y^1)^2 - 4y^0y^2$ of the automorphism subgroup $\{B_0, B_1, B_2\}$. Hence there are more nonconjugate subalgebras of the algebra $\{Y_1, Y_p, Y_{p^2}\}$ than it is pointed out in Theorem 1. However, for our purpose it is important to use the infinite set of similarities from Theorem 1 and we obtain only 3 subalgebras.

4. Optimal systems of complements to the Abelian ideal

Each of the algebras with $N=9$, 11; 4, 14, and $N=15$ decomposes in the semidirect sum of the Abelian ideal J_6 and the subalgebra complement. As the first step of enumeration of subalgebras it is necessary to consider its projections on the complement.

We use the optimal system of the subalgebra H to construct the submodel. It is important to know what symmetries of the big model are admitted by the submodel. It turns out that the submodel admits the factor algebra $N_L(H)/H$, where $N_L(H)$ is the normalizer of the subalgebra H in the algebra L. Therefore the optimal system must contain the normalizers of the corresponding subalgebra. We call this optimal system *the normalized optimal system.*

For $N = 9$ the subspace $\{Y_1\}$ is the center. Let the set of automorphisms for L_{11} be equal to the set of automorphisms for $L_{11} \oplus \{Y_1\}$ and let the projections of the subalgebra be equal to 0. Then the optimal system coincides with the optimal system for L_{11}.

Let us consider the complement subalgebra to the Abelian ideal J_6 and assume that projections on this subalgebra do not vanish. The projection on the complement subalgebra decomposes in the direct sum $J_3 \oplus \{10, 11\} \oplus \{\bar{1}\}$ of the ideals. The automorphism group decomposes in the semidirect product of the invariant subgroup acting on the Abelian ideal and the subgroup acting on the complement: $\{S, A_{10}, A_{11}, \epsilon_2\}$. The last subgroup decomposes in the direct product of the invariant subgroups acting on the ideals J_3 and $\{10, 11\}$.

It is known that the rotational group S transforms the subalgebras of the simple algebra J_3 to one and only one nonzero proper subalgebra $\{X_7\}$. The transformations A_{10} and A_{11} give two nonzero proper subalgebras $\{X_{10}\}$ and $\{X_{11}\}$. Now we can calculate the normalized optimal system of the complement subalgebras. We have formulated the results as Table 5.

Table 5.

r	i	Basis	Index of Normalizer
6	1	$7, 8, 9, 10, 11, \bar{1}$	=6.1
5	1	$7, 8, 9, 11, \bar{1}$	=5.1
	6	$7, 8, 9, 10, a11 + \bar{1}$	6.1
4	8	$7, 1, 9, a11 + \bar{1}; a \neq 0$	5.1
	2	$7, 8, 9, 10 + \bar{1}$	5.2, a=0
	3	$7, 8, 9, \bar{1}$	6.1
	4	$7, 10, 11, \bar{1}$	=4.4
3	1	$7, 15, \bar{1}$	=3.1
	2	$5 + a11, 10, b11 + \bar{1}$	8.4
	3	$10, 11, \bar{1}$	6.1
2	1	$7 + b11, a11 + \bar{1}; a^2 + b^2 \neq 0$	3.1
	2	$7 + b10, 10 + \bar{1}$	3.2, a=b=0
	3	$7, \bar{1}$	4.4
	4	$7 + 10, \bar{1}$	4.4
	5	$11, \bar{1}$	5.1
	6	$10, a11 + \bar{1}$	6.1

To be continued

(Continuation of Table 5)			
r	i	Basis	Index of Normalizer
1	1	$a7 + b11 + \bar{1}, a \neq 0, b \neq 0$	3.1
	2	$a7 + 10 + \bar{1}, a \neq 0$	3.2; a=b=0
	3	$a7 + \bar{1}, a \neq 0$	4.4
	4	$b11 + \bar{1}, b \neq 0$	5.1
	5	$10 + \bar{1}$	5.2, a=0
	6	$\bar{1}$	6.1

In the first column of Table 5 r is the dimension of the subalgebra. In the second column of Table 5 i is the number of the subalgebra for the given dimension. In the third column of Table 5 there are sets of operators which are invariants of the automorphism group. For example, $a7 + b11 + \bar{1} = aX_7 + bX_{11} + Y_1$ where a and b are constants. If there is an index $r.i$ in the line L of the fourth column of Table 5 then it means that the subalgebra with the dimension r and the number i from Table 5 is the normalizer of the subalgebra from the line L of Table 5. The symbol "=" means the selfnormalized subalgebra.

For $N = 11$ the subspace $\{Y_1, Y_p\}$ is an ideal. The 1-dimensional subalgebras of the ideal are conjugated only with two subalgebras Y_1 and Y_p. By Theorem 1, the subalgebra Y_1 is similar to Y_p. If the projection of a subalgebra on $\{Y_1, Y_p\}$ is equal to 0 or if it is 1-dimensional then the subalgebra is conjugated with the optimal system for L_{11} or with the optimal system for $L_{11} \oplus \{Y_1\}$. It remains to construct the optimal system for the subalgebra with a full projection on $\{Y_1, Y_p\}$.

For $N = 11$ we have constructed the optimal system of the projections on the complement subalgebra to J_6 (see Table 6).

Table 6.

r	i	Basis	Index of Normalizer
7	1	7,8,9,10,11,$\bar{1}, p$	=7.1
6	1	7,8,9,11,$\bar{1}, p$	=6.1
	2	$7, 8, 9, 10, \bar{1}, p$	7.1
5	1	7,8,9,$\bar{1}, 10 + p$	6.6
	2	$7, 8, 9, \bar{1}, a11 + p, a \neq 0$	6.1
	5	$7, 8, 9, 10 + \bar{1}, 11 + p$	=5.3
	4	$7, 8, 9, \bar{1}, p$	7.1
			To be continued

(Continuation of Table 6)			
r	i	Basis	Index of Normalizer
	5	$7, 10, 11, \bar{1}, p$	=5.5
4	1	$7, 11, \bar{1}, p$	=4.1
	2	$7 + a11, 10, \bar{1}, b11 + p$	5.5
	3	$10, 11, \bar{1}, p$	7.1
3	1	$7 + a11, \bar{1}, b11 + p; a^2 + b^2 \neq 0$	4.1
	2	$7, \bar{1}, p$	5.5
	3	$7, 10 + \bar{1}, 11 + p$	=3.3
	4	$7 + a10, \bar{1}, b10 + p; a^2 + b^2 = 1$	4.2;a=b=0
	5	$10, \bar{1}, p$	6.1
	6	$10, \bar{1}, a11 + p$	7.1
2	1	$10 + \bar{1}, a7 + 11 + p; a \neq 0$	3.3
	2	$\bar{1}, a7 + b11 + p; a \neq 0, b \neq 0$	4.1
	3	$\bar{1}, a7 + 10 + p; a \neq 0$	5.5
	4	$\bar{1}, a7 + p; a \neq 0$	5.5
	5	$10 + \bar{1}, 11 + p$	5.3
	6	$\bar{1}, a11 + p; a \neq 0$	6.1
	7	$\bar{1}, 10 + p$	7.1
	8	$\bar{1}$	7.1

For $N = 4$ the subspace – subalgebra $\{12\}$ is the complement of the ideal L_{11}. If the projection on this subspace is equal to 0 then the automorphisms are inconsistent with the automorphisms of L_{11} and the optimal system of the subalgebra coincide with the optimal system for L_{11}. The subalgebras with the nonvanishing projections on the subspace $\{12\}$ are studied in [3].

In Table 7 the optimal system of the projections on the complement subalgebra to the Abelian ideal J_6 is constructed on the basis of the decomposition $J_3 \oplus (\{10\} \dot{\oplus} \{11, 12\})$ with the help of the automorphisms S, A_{10}, A_{11}, A_{12}, ϵ_2.

Table 7.

r	i	Basis	Index of Normalizer
6	1	$7, 8, 9, 10, 11, 12$	=6.1
5	1	$7, 8, 9, 11, 12$	=5.1
	2	$7, 8, 9, 10, a11 + 12$	6.1
4	1	$7, 8, 9, a11 + 12; a \neq -1$	5.1
	2	$7, 8, 9, -11 + 12$	6.1
			To be continued

(Continuation of Table 7)			
r	i	Basis	Index of Normalizer
	3	$7, 8, 9, 10 - 11 + 12$	5.2, a=-1
	4	$7, 10, 11, 12$	=4.4
3	1	$7, 11, 12$	=3.1
	2	$7 + a11, 10, b11 + 12$	4.4
	3	$7 + a12, 10, 11; a \neq 0$	4.4
	4	$10, 11, 12$	6.1
2	1	$7 + a11, b11 + 12; a^2 + (b + 1)^2 \neq 0$	3.1
	2	$7 + a12, 11; a \neq 0$	3.1
	3	$7, -11 + 12$	4.4
	4	$7 + a10, b10 - 11 + 12; a^2 + b^2 = 1$	3.2;a=-1,b=0
	5	$3 + a11 + b12, 10; b \neq 0$	4.4
	6	$11, 12$	5.1
	7	$10, a11 + 12$	6.1
1	1	$a7 + 10 - 11 + 12; a \neq 0$	3.2;a=-1
	2	$10 - 11 + 12$	5.2; a=-1
	3	$a7 + b11 + 12; a \neq 0, b \neq -1$	3.1
	4	$a7 - 11 + 12, a \neq 0$	4.4
	5	$b11 + 12; b \neq -1$	5.1
	6	$-11 + 12$	6.1

For the subalgebras with $N = 14$ and $N = 15$ it is sufficient to consider the subalgebras with the full projections on $\{\bar{1}, p\}$ and $\{\bar{1}, p, p^2\}$ respectively if the similarity transformations from Theorem 1 are taken into account. For $N = 14$ the projection on $\{12\}$ does not vanish. In Tables 8 and 9 there are optimal systems of the projections on the complement subalgebra to J_6.

Table 8.

r	i	Basis	Index of Normalizer
8	1	$7, 8, 9, 10, 11, 12, \bar{1}, p$	$= 8.1$
7	1	$7, 8, 9, 11, 12, \bar{1}, p$	=7.1
	2	$7, 8, 9, 10, a11 + 12, \bar{1}, b11 + p$	8.1
6	1	$7, 8, 9, a11 + 12, \bar{1}, b11 + p; (a + 1)^2 + b^2 \neq 0$	7.1
	2	$7, 8, 9, -11 + 12, \bar{1}, p$	8.1
			To be continued

(Continuation of Table 8)

r	i	Basis	Index of Normalizer
	3	$7,8,9,11+\bar{1},a12+p;a\neq 0$	7.1
	4	$7,8,9,a10-11+12,\bar{1},b10+p;$	7.2;
			$b=0$
		$a^2+b^2=1$	$a=-1$
	5	$7,8,9,-11+12,10+\bar{1},11+p$	=6.5
	6	$7,8,9,10,\bar{1},a11+b12+p;b\neq 0$	8.1
	7	$7,10,11,12,\bar{1},p$	=6.7
5	1	$7,8,9,\bar{1},a11+b12+p;b\neq 0,a+b\neq 0$	7.1
	2	$7,8,9,\bar{1},a(-11+12)+p;a\neq 0$	8.1
	3	$7,8,9,10+\bar{1},(1-a)11+a12+p;a\neq 0$	=5.3
	4	$7,8,9,\bar{1},10+a(-11+12)+p;a\neq 0$	7.2;
			$b=0$
			$a=-1$
	5	$7,11,12,\bar{1},p$	=5.5
	6	$7+a11,10,b11+12,\bar{1},c11+p$	6.7
	7	$7+a12,10,11,\bar{1},b12+p;a^2+b^2\neq 0$	6.7
	8	$10,11,12,\bar{1},a7+p;a\neq 0$	6.7
	9	$10,11,12,\bar{1},p$	8.1
4	1	$7+a11,b11+12,\bar{1},c11+p;a^2+c^2\neq 0$	5.5
	2	$7,-11+12,\bar{1},p$	6.7
	3	$7+a11,10-11+12,\bar{1},p;a\neq 0$	5.5
	4	$7,-11+12,10+\bar{1},11+p$	=4.4
	5	$7+a10,b10-11+12,\bar{1},c10+p;$	5.6;
		$a^2+b^2+c^2=1$	$b=-1$
			$a=c=0$
	6	$7+a11+b12,10,\bar{1},c11+d12+p;$	6.7
		$b^2+d^2\neq 0$	
	7	$11,12,\bar{1},a7+p;a\neq 0$	5.5
	8	$10,a11+12,\bar{1},b7+c11+p;b\neq 0$	6.7
	9	$10,11,\bar{1},a7+b12+p;b\neq 0,a\neq 0$	6.7
	10	$11,12,\bar{1},p$	7.1
	11	$10,a11+12,\bar{1},b11+p$	8.1
	12	$10,11,\bar{1},a12+p;a\neq 0$	8.1
3	1	$7+a11+b12,\bar{1},c11+d12+p$	5.5
		$c+d\neq 0\vee a+b\neq 0$	

To be continued

(Continuation of Table 8)			
r	i	Basis	Index of Normalizer
	2	$7+a(-11+12), 10+\bar{1}, (1-b)11+b12+p$ $a^2+b^2\neq 0$	4.4
	3	$7, a(-11+12), \bar{1}, b(-11+12)+p$	6.7
	4	$7+a10+b(-11+12),\ \bar{1},$ $c10+d(-11+12)+p;\ a^2+c^2=1$	5.6; $b=-1$ $a=b=0$
	5	$10-11+12, \bar{1}, a7+b10+p;\ a\neq 0$	5.6; $b=-1$ $a=c=0$
	6	$-11+12, \bar{1}, a7+10+p;\ a\neq 0$	5.6; $b=-1$ $a=c=0$
	7	$a11+12,\ \bar{1},\ b7+c11+p;\ b\neq 0,$ $(a+1)^2+c^2\neq 0$	5.5
	8	$-11+12, \bar{1}, a7+p; a\neq 0$	6.7
	9	$-11+12, 10+\bar{1}, a7+11+p; a\neq 0$	4.4
	10	$11, \bar{1}, a7+b12+p; a\neq 0, b\neq 0$	5.5
	11	$10, \bar{1}, a7+b11+c12+p; a\neq 0, c\neq 0$	6.7
	12	$a11+12, \bar{1}, b11+p; (a+1)^2+b^2\neq 0$	7.1
	13	$-11+12, \bar{1}, p$	8.1
	14	$-11+12, 10+\bar{1}, 11+p$	6.5
	15	$10-11+12,\ \bar{1}, a10+p$	7.2; $a=-1$
	16	$-11+12,\ \bar{1}, 10+p$	7.2; $a=-1$
	17	$11, \bar{1}, a12+p; a\neq 0$	7.1
	18	$10, \bar{1}, a11+b12+p; b\neq 0$	8.1
2	1	$\bar{1}, a7+b11+c12+p; a\neq 0, b+c\neq 0$	5.5
	2	$\bar{1}, a7+b(-11+12)+p; a\neq 0$	6.7
	3	$\bar{1},\ a7+10+b(-11+12)+p;\ a\neq 0$	5.6; $b=-1$ $a=c=0$
	4	$10+\bar{1}, a7+(1-b)11+b12+p; a\neq 0, b\neq 0$	4.4
	5	$\bar{1}, a11+b12+p; a+b\neq 0$	7.1
	6	$\bar{1}, a(-11+12)+p$	8.1
	7	$\bar{1},\ 10+a(-11+12)+p$	7.2; $a=-1$
			To be continued

(Continuation of Table 8)			
r	i	Basis	Index of Normalizer
	8	$10+\bar{1},(1-a)11+a12+p;a\neq 0$	6.5

Table 9.

r	i	Basis	Index of Normalizer
9	1	$7,8,9,10,11,12,\bar{1},p,p^2$	=9.1
8	1	$7,8,9,11,12,\bar{1},p,p^2$	=8.1
	2	$7,8,9,10,a11+b12,\bar{1},p,p^2;a^2+b^2=1$	9.1
7	1	$7,8,9,a11+b12,\bar{1},p,p^2;a^2+b^2=1$	8.1
	2	$7,8,9,-11+12,\bar{1},p,p^2$	9.1
	3	$7,8,9,10-11+12,\bar{1},p,p^2$	8.2;$b=0$, $a=-1$
	4	$7,8,9,10,\bar{1},p,p^2$	9.1
	5	$7,10,11,12,\bar{1},p,p^2$	=7.5
6	1	$7,8,9,\bar{1},p,p^2$	9.1
	2	$7,11,12,\bar{1},p,p^2$	6.2
	3	$7+a11,10,b11+12,\bar{1},p,p^2$	7.5
	4	$7+a12,10,11,\bar{1},p,p^2$	7.5
	5	$10,11,12,\bar{1},p,p^2$	9.1
5	1	$7+a11,b11+12,\bar{1},p,p^2;a^2+(b+1)\neq 0$	6.2
	2	$7+a12,11,\bar{1},p,p^2$	6.2
	3	$7,-11+12,\bar{1},p,p^2$	7.5
	4	$7+a10,b10-11+12,\bar{1},p,p^2;$ $a^2+b^2=1$	6.3;$a=0$, $b=-1$
	5	$7+a11+b12,10,\bar{1},p,p^2$	7.5
	6	$11,12,\bar{1},p,p^2$	8.1
	7	$10,a11+b12,\bar{1},p,p^2;a^2+b^2=1$	9.1
4	1	$7+a11+b12,\bar{1},p,p^2;a+b\neq 0$	6.2
	2	$7+a(-11+12),\bar{1},p,p^2$	7.5
	3	$7+10+a(-11+12),\bar{1},p,p^2$	6.3;$a=0$, $b=-1$
	4	$a11+b12,\bar{1},p,p^2;a^2+b^2=1$	8.1
			To be continued

(Continuation of Table 9)			
r	i	Basis	Index of Normalizer
	5	$-11+12, \bar{1}, p, p^2$	9.1
	6	$10-11+12, \bar{1}, p, p^2$	8.2;$b=1$ $a=-1$
	7	$10, \bar{1}, p, p^2$	9.1
3	1	$\bar{1}, p, p^2$	9.1

5. Conclusion

For the algebras with $N = 9$, 11; 4, 14, and $N = 15$ we calculate the stationary subgroup at the second stage of the subalgebras enumeration. For all projections from Tables 5–9 we also calculate an invariant subgroup on J_6. Projections on J_6 are classified with the help of the stationary subgroup and a change of the basis. In the class of the conjugate subalgebras there is a subalgebra such that it may be characterized by means of the least number of parameters. We call it the simplest subalgebra. Parameters corresponding to the simplest subalgebra are the invariants of the stationary subgroup. The optimal systems obtained by this way are enlarged many times compared with the systems obtained in [1–5].

References

1. Ovsyannikov, L. V. (1994): The Program SUBMODELS. Gas Dynamics, Applied Mathematics and Mechanics, **58**, No. 4, 30–55.
2. Cherevko, A. A. (1996): *Optimal System of Subalgebras for Lie Algebra of Operators Admitted by the System of the Gasdynamics Equations with the State Equation* $p = f(S)\rho^{5/3}$, Preprint No. 4-96, Institute of Hydrodynamics, Novosibirsk, 39 p.
3. Golovin, S. V. (1996): *Optimal System of Subalgebras for Lie Algebra of Operators Admitted the Gasdynamics Equations in the Case of the Polytropic Gas*, Preprint No. 5-96, Institute of Hydrodynamics, Novosibirsk, 31 p.
4. Khabirov, S. V. (1998): *The Optimal System of Subalgebras Admitted the Gasdynamics Equations*, Preprint, Print, Ufa, 33 p.
5. Patera, J., Shap, R. T., Winternitz, P., and Zassenhaus, H. (1977): Continuous subgroups of the fundamental groups of Physics. III. The De Sitter groups, J. Math. Phys., **18**, No. 12, 2259–2288.