Selçuk J. Appl. Math.
Vol. 4, No. 1, pp. 42–57, 2003

Selçuk Journal of Applied Mathematics

Graphics Constructor 2.0

Ayşe Bulgak[1] **and Diliaver Eminov**[2]

[1] Research Centre of Applied Mathematics, Selçuk University, Konya, Turkey; e-mail: `abulgak@selcuk.edu.tr`
[2] Spectel Ltd., Dublin, Ireland; e-mail: `Diliaver.Eminov@spectel.com`

Received: April 20, 2003

Summary. An experimental version of computer-aided methods of teaching mathematical analysis has been elaborated in the Research Centre of Applied Mathematics of the Selçuk University. The main part of this version is a textbook "Analysis"[1]. It is supported by the computer dialogue program Graphics Constructor [4] and its extended version Graphics Constructor 2.0.

Key words: spline, piecewise polynomial, computer dialogue program

2000 Mathematics Subject Classification: 65D07, 97U070

1. Introduction

During past few years teachers and computer engineers have been focusing attention upon computer-aided methods in education. See, for example, the following books [5], [7], [8], [12] published recently by Springer. At present, there are textbooks including theory of functions of one real variable, elements numeric analysis and methods of solving practical problems (see, for instance, [5]). Such books are intended for a wide range of readers, including students, who desire to specialize in the field of mathematical analysis in future. Teachers have been trying to find out what and how to teach in Mathematics. Though secondary school programs are extremely overloaded, some first-year students have a very little knowledge of elementary functions and, sometimes, have difficulties working with expressions like $(a-b)^2$ or $a^2 - b^2$ (see introduction to [11]).

One of the basic concepts of numerical analysis is polynomial. In our opinion, it wouldn't be exaggeration to say that 19th century has gone by under a sign of polynomial. But by the end of 20th century matrices and systems of linear equations gradually forced out polynomial due to a rapid development of computers. In particular, even an apparently simple task of evaluation a high degree polynomial with guaranteed accuracy at a given point requires solving a system of linear algebraic equations (see, [10]). Probably this fact is one of the reasons why the scientists have been focusing attention upon first, second and third-degree splines (see, for example, [6], [9], [12], [14], [16]). We suggest teaching the basic concepts of mathematical analysis using polynomials and splines up to the third degree. Though time-frame of educational process is limited, the elaborated course of mathematical analysis allows acquainting students, even with a weak understanding of trigonometry and other elementary functions, with basic properties of functions of one real variable. It then builds a basis for further, deeper learning of Mathematics.

Several programming languages, such as $MATLAB$, $MAPLE$, $MATHEMATICA$ that have been created during last 10-15 years, are successfully being used in the educational process. Many books were devoted to studying of that languages and Mathematics using these languages (see, for example, [7], [8], [13], [15]). Such books are mostly useful for graduate and postgraduate students. As Mathematics is, in essence, a language itself, we come to understanding that it is less efficient teaching it to undergraduate students by using other languages like $MATLAB$, $MAPLE$, $MATHEMATICA$.

In the Center of Applied Mathematics of Selcuk University (Konya, Turkey), a set of simple computer dialog-based applications is suggested as a possible solution to above-mentioned dilemma. These applications are as difficult in use as a pocket calculator and, therefore, not requiring a special learning. One of the applications, Graphics Constructor 2.0 (GC2), is described further in this article.

The application is intended for undergraduate students. It allows using computer in teaching of such subjects as introduction in mathematical analysis, introduction in differential equations and theory of approximation. The design of the application is to do chore in drawing graphics of real functions and computing approximations of definite integrals. It is available at http://www.sumam.selcuk.edu.tr/software.html.

GC2 is a dialog-based application that works under Windows-95 and higher, using intuitive dialogs of OS Windows. It is written on MS Visual C++.

GC2 allows working with Lagrangian interpolating polynomials (Section 2), polygons (Section 3), quadratic (Section 4) and cubic splines (Section 5). Some geometrical applications of definite integrals are given in Section 3. A theorem of parametrization of quadratic spline via its slope at an initial point is recalled in Section 4. In this section formulas for computing integrals of quadratic splines on an interval $[x_0, x_{n-1}]$ can be found. A theorem of parametrization of cubic spline via its slopes at the nodes is recalled in the section 5. In this section formulas for computing integrals of cubic splines on an interval $[x_0, x_{n-1}]$ can be found. GC2 is described in outline in Section 6. Some its applications are given in Sections 7 and 8.

2. Lagrangian interpolating polynomial

Start with Table 1.

X	x_0	x_1	x_2	$\dots$	x_{n-2}	x_{n-1}
Y	y_0	y_1	y_2	$\dots$	y_{n-2}	y_{n-1}

Table 1

Here $x_0 < x_1 < \ldots < x_{n-1}$ are distinct ordered real numbers. A polynomial $g(x)$, for which $y_i = g(x_i)$, is said to interpolate the given table at the nodes x_i ($0 \leq i \leq n-1$).

$$g(x) = \sum_{i=0}^{n-1} y_i L_i(x), L_i(x) = \prod_{k=0, k \neq i}^{n-1} \frac{x - x_k}{x_i - x_k}.$$

It is known as *Lagrangian interpolating polynomial* [1].

3. Polygonal spline

Return to Table 1. The function

$$g(x) = a_k x + b_k, x_{k-1} \leq x \leq x_k,$$

$$a_k = \frac{y_k - y_{k-1}}{x_k - x_{k-1}}, b_k = y_k - a_k x_k, k = 1, 2, \ldots, n-1$$

is known as a polygonal spline for the given table. It is clear that $y_i = g(x_i)$ at the nodes x_i $(0 \leq i \leq n-1)$. Obviously, there exists a unique polygonal spline which satisfies to Table 1.

The definite integral $\int_{x_0}^{x_{n-1}} g(x)dx$ can be computed by the formula

$$P = \sum_{i=1}^{n-1} \frac{y_i + y_{i-1}}{2}(x_i - x_{i-1}).$$

This formula is known as a trapezoid rule for numerical integration.

A length of a curve $y = g(x)$, $x_0 \le x \le x_{n-1}$ can be computed by the formula

$$L = \int_{x_0}^{x_{n-1}} \sqrt{1 + g'^2(x)}dx = \sum_{i=1}^{n-1} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}.$$

An area of a solid obtained by rotating $y = g(x)$ between $x = x_0$, $x = x_{n-1}$ around x-axis is given as follows

$$S = \int_{x_0}^{x_{n-1}} g(x)\sqrt{1 + g'^2(x)}dx = \sum_{i=1}^{n-1} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}.$$

It can be computed by the formula

$$S = 2\pi \sum_{i=1}^{n-1} S_i.$$

Introduce the notation, $h_i = |y_i|$,$i = 0, 1, \ldots, n-1$.

In the case of $y_{i-1}y_i < 0$ to find S_i, $i = 1, 2, \ldots, n-1$, we use the formulas $S_i = S_{i,1} + S_{i,2}$, where

$$S_{i,1} = \sqrt{1 + a_{i,1}^2}(\frac{a_{i,1}}{2}(z_i + x_{i-1}) + b_{i,1})(z_i - x_{i-1}),$$

$$S_{i,2} = \sqrt{1 + a_{i,2}^2}(\frac{a_{i,2}}{2}(x_i + z_i) + b_{i,2})(x_i - z_i)$$

and

$$z_i = \frac{x_i y_{i-1} - x_{i-1} y_i}{y_i - y_{i-1}}, a_{i,1} = -\frac{h_{i-1}}{z_i - x_{i-1}}, b_{i,1} = h_{i-1} - a_{i,1}x_{i-1},$$

$$a_{i,2} = -\frac{h_i}{x_i - z_i}, b_{i,2} = h_i - a_{i,2}z_i.$$

In the case of $y_{i-1}y_i \ge 0$, we have

$$S_i = \sqrt{1 + a_{i,1}^2}(\frac{a_{i,1}}{2}(x_i + x_{i-1}) + b_{i,1})(x_i - x_{i-1}),$$

where

$$a_{i,1} = -\frac{h_i - h_{i-1}}{x_i - x_{i-1}}, b_{i,1} = h_{i-1} - a_{i,1}x_{i-1}.$$

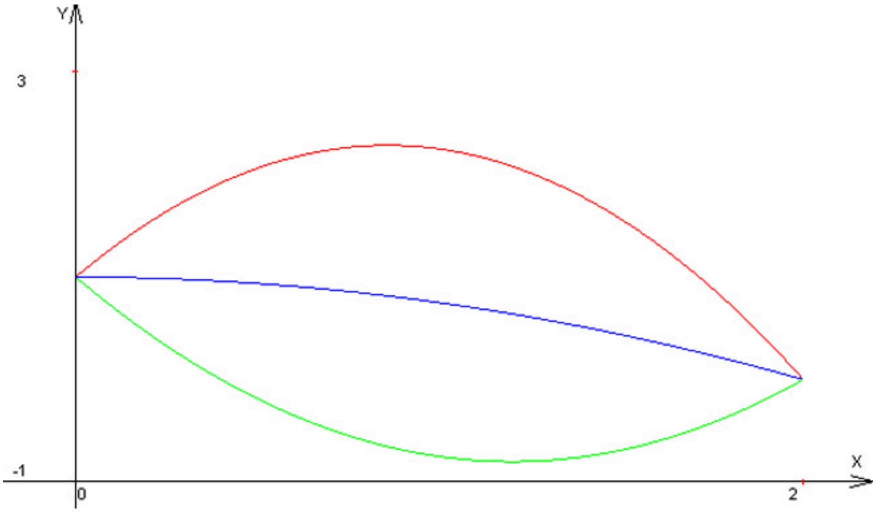

Fig. 1.

The volume of a solid obtained by rotating $y = g(x)$ between $x = x_0$, $x = x_{n-1}$ around the x-axis is given as follows

$$V = \pi \int_{x_0}^{x_{n-1}} g^2(x)dx.$$

It is computed as:

$$V = \pi \sum_{i=0}^{n-1} \frac{a_i^2}{3}((x_i^3 - x_{i-1}^3) + a_i b_i (x_i^2 - x_{i-1}^2) + b_i^2(x_i - x_{i-1})),$$

where

$$a_i = \frac{y_i - y_{i-1}}{x_i - x_{i-1}}, b_i = y_{i-1} - a_i x_{i-1}, i = 1, 2, \ldots, n-1.$$

4. Quadratic spline

It is clear that many more quadratic splines (the second-degree splines) satisfy to Table 1. For example, Table 2 defines three different quadratic splines whose graphics are given in Fig. 1.

X	0	2
Y	1	0

Table 2

As another example, consider Table 3.

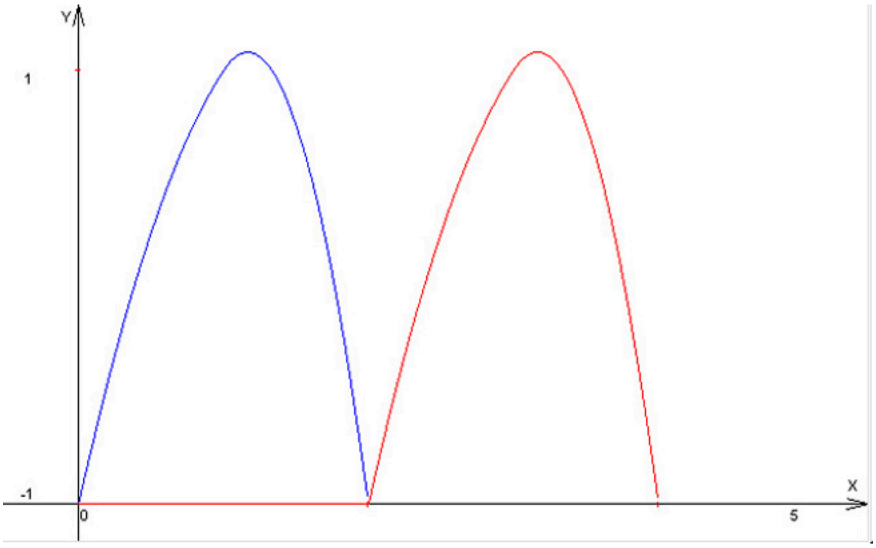

Fig. 2.

X	0	1	2	3	4
Y	-1	1	-1	1	-1

Table 3

The graphics of a quadratic spline, which corresponds to the table, is given in Fig. 2. This quadratic spline has no derivative in $x = -1$.

Following theorem can be easily verified.

Theorem 1. *A quadratic spline $y = f(x) \in C^1[x_0, x_{n-1}]$, which satisfies Table 1 and the condition $f'(x_0) = F_0$, is unique. Here F_0 is a real parameter. The function $f(x)$ has following representation*

$$f(x) = a_j x^2 + b_j x + c_j, x_{j-1} \le x < x_j, j = 1, 2, \dots, n-1,$$

where

$$a_1 = \frac{1}{x_1 - x_0}\left(\frac{y_1 - y_0}{x_1 - x_0} - F_0\right), b_1 = F_0 - 2a_1 x_0, c_1 = y_0 - a_1 x_0^2 - b_1 x_0,$$

$$a_j = \frac{1}{x_j - x_{j-1}}\left(\frac{y_j - y_{j-1}}{x_j - x_{j-1}} - 2a_{j-1}x_{j-1} - b_{j-1}\right),$$

$$b_j = b_{j-1} + 2(a_{j-1} - a_j)x_{j-1}, c_j = y_{j-1} - a_j x_{j-1}^2 - b_j x_{j-1}.$$

Hence, instead of Table 1 we can use Table 4.

X	x_0	x_1	x_2	$\dots$	x_{n-2}	x_{n-1}
Y	y_0	y_1	y_2	$\dots$	y_{n-2}	y_{n-1}
F	F_0			$\dots$		

Table 4

The definite integral $\int_{x_0}^{x_{n-1}} f(x)dx$ can be computed as follows

$$\int_{x_0}^{x_{n-1}} f(x)dx = \sum_{i=0}^{n-1}(x_i - x_{i-1})(\frac{a_i}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2) + \frac{b_i}{2}(x_i + x_{i-1}) + c_i).$$

The coefficients a_j , b_j, c_j, $j = 1, 2, \ldots, n-1$, are defined by Theorem 1.

5. Cubic spline

Take Table 5 with a real sequence $F_0, F_1, \ldots, F_{n-1}$.

X	x_0	x_1	x_2	$\ldots$	x_{n-2}	x_{n-1}
Y	y_0	y_1	y_2	$\ldots$	y_{n-2}	y_{n-1}
F	F_0	F_1	F_2	$\ldots$	F_{n-2}	F_{n-1}

Table 5

Following theorem is true (see, for example, [16])

Theorem 2. *A cubic spline $y = h(x) \in C^1[x_0, x_{n-1}]$, which satisfies Table 5 and the conditions $f'(x_j) = F_j$, $j = 0, 1, \ldots, n-1$, is unique. The function $h(x)$ has following representation*

$$h(x) = a_i x^3 + b_i x^2 + c_i x + d_i,$$

$$x_{i-1} \le x < x_i, i = 1, 2, \ldots, n-1,$$

where

$$w_i = \frac{1}{x_i - x_{i-1}}(\frac{y_i - y_{i-1}}{x_i - x_{i-1}} - F_{i-1}), i = 1, 2, \ldots, n-1,$$

$$a_i = \frac{1}{x_i - x_{i-1}}(\frac{F_i - F_{i-1}}{x_i - x_{i-1}} - 2w_i); b_i = -(x_i + 2x_{i-1})a_i + w_i;$$

$$c_i = f_{i-1} - 3a_i x_{i-1}^2 - 2b_i x_{i-1}; d_i = y_{i-1} - a_i x_{i-1}^3 - b_i x_{i-1}^2 - c_i x_{i-1}.$$

The definite integral $H = \int_{x_0}^{x_{n-1}} h(x)dx$ for the cubic (third-degree) spline h(x) can be computed as follows

$$H = \sum_{i=1}^{n-1}(x_i - x_{i-1})(\frac{a_i}{4}(x_i^3 + x_i^2 x_{i-1} + x_i x_{i-1}^2 + x_{i-1}^3)$$

$$+\frac{b_i}{3}(x_i^2 + x_i x_{i-1} + x_{i-1}^2) + \frac{c_i}{2}(x_i + x_{i-1}) + d_i).$$

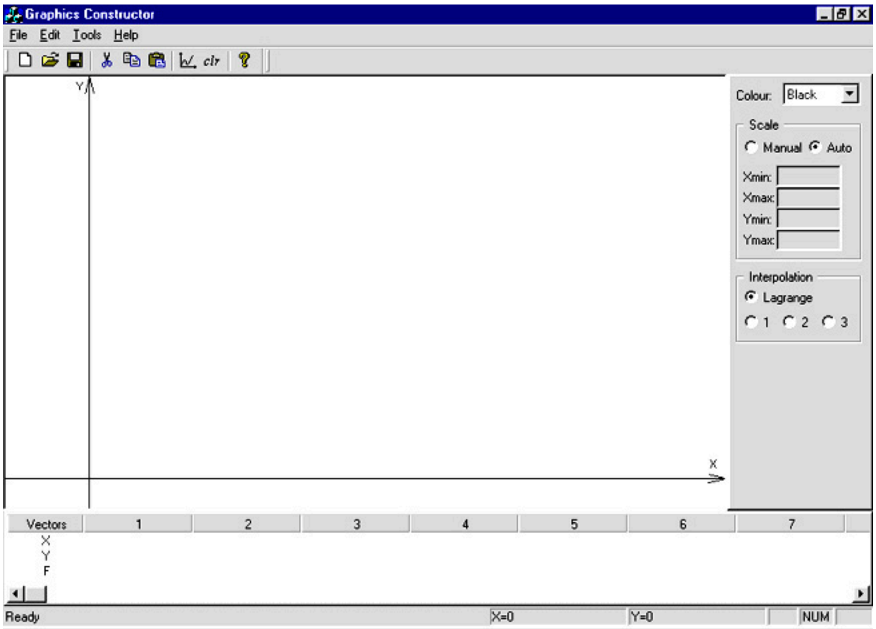

Fig. 3.

6. Graphics Constructor 2.0

The main window of GC2 comprises of the graphical area, main menu with a toolbar, control panel and status bar (see Fig. 3).

The main menu, toolbar and status bar purpose is as common and intuitive as any Windows application.

6.1. The main menu

The main menu includes four submenus - File (see Table 6), Edit, Tools (see Table 7) and Help (see Table 8).

New	Open a new worksheet
Open Vectors	Load Vectors table from file
Save Vectors As...	Save Vectors table to file
Save Picture As...	The graphics is saved in the graphical format .bmp
Exit	Exit the program

Table 6

Run	Draw graphics
Clear	Clear graphics area. Does not change Vectors table
SetArea	Initiate selecting new scale values

Table 7

About	information about authors of the program

Table 8

Menu "Edit" includes submenus undo, cut, copy, paste that can be used during editing elements of Table "Vectors".

6.2. Table "Vectors"

Table "Vectors" contains three rows X, Y, F, where elements are numbered from 0 to 300.

Order of filling in the table is not important. Note that the sequence X must be increasing only. "Tab" button allows moving to the next right cell and "Shift+Tab" button combination to the left cell. Any cell can be selected by using mouse.

Each value has the form $[-]d.dddde[sign]ddd$ where d is a single decimal digit, $dddd$ is one or more decimal digits, ddd is up to three decimal digits, and sign is + or −.

6.3. Control panel

Control panel contains three parts : "Colour", "Interpolation" and "Scale"

"Colour" is a combo-box that allows choosing graphics colour. It has six predefined colours - black, blue, red, green, yellow and purple.

"Interpolation" has four alternatives to choose: from 1 to 3 and "Lagrange ". The numbers correspond to the first-degree, quadratic and cubic splines respectively. "Lagrange" corresponds to Lagrangian interpolating polynomial.

"Scale" allows choosing a rectangle where the graphics will be drawn. This rectangle is defined by values of $Xmax$, $Xmin$, $Ymax$, $Ymin$.

"Scale" has two modes "Manual" and "Auto".

If "Auto" is chosen, the $Xmax$, $Xmin$, $Ymax$, $Ymin$ are inactive and the program computes theirs by formulas

$$Xmin = x_0, Xmax = x_{n-1},$$

$$Ymin = \min_{j=0,1,\ldots,n-1} y_j, Ymax = \max_{j=0,1,\ldots,n-1} y_j.$$

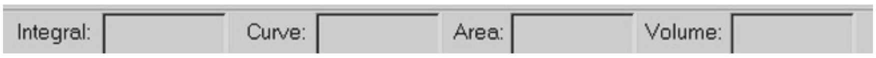

Fig. 4.

Here $n, x_0, x_1, \ldots, x_{n-1}, y_0, y_1, \ldots, y_{n-1}$ are taken from table "Vectors".

If "Manual" is chosen, $Xmax$, $Xmin$, $Ymax$, $Ymin$ have to be entered manually and $Xmax > Xmin, Ymax > Ymin$. These reals have to be entered in the same format as the cells of Table "Vectors".

If "Set Area" in menu "Tools" is chosen, then a scaling rectangle can be drawn by using the mouse one. This activates "Manual" mode and sets corresponding $Xmax$, $Xmin$, $Ymax$, $Ymin$.

Note. If elements of row F are absent, it means that they equal to 0.

6.4. Results bar

Existence of a result bar depends on "Interpolation" regime. The result bar doesn't exist in the case of regime "Lagrange". The result bar contains four boxes which are headed as "Integral", "Curve", "Area" and "Volume" (see Fig. 4) in the case of regime "1" (polygonal spline) and contains only one box which is headed as "Integral" in the case of regime "2" or "3".

If the program computes a polygonal spline, it also computes values

$$\int_{x_0}^{x_{n-1}} g(x)dx, L = \int_{x_0}^{x_{n-1}} \sqrt{1 + (g'(x))^2}dx,$$

$$S = 2\pi \int_{x_0}^{x_{n-1}} g(x)\sqrt{1 + (g'(x))^2}dx, V = \pi \int_{x_0}^{x_{n-1}} g^2(x)dx$$

Here $g(x)$ is a computed first-degree spline. One can read the computed values on boxes "Integral", "Curve", "Area" and "Volume" respectively.

If the program computes a second or third degree spline, it also computes a value of definite integral of the corresponding spline on interval $[x_0, x_{n-1}]$. One can read the computed value on box "Integral".

6.5. Graphical area

The graphics area is organized by orthogonal co-ordinate system, x-axis and y-axis. These lines are black. The crossing point corresponds to the point $(Xmin, Ymin)$. The points $(Xmin, Ymax)$

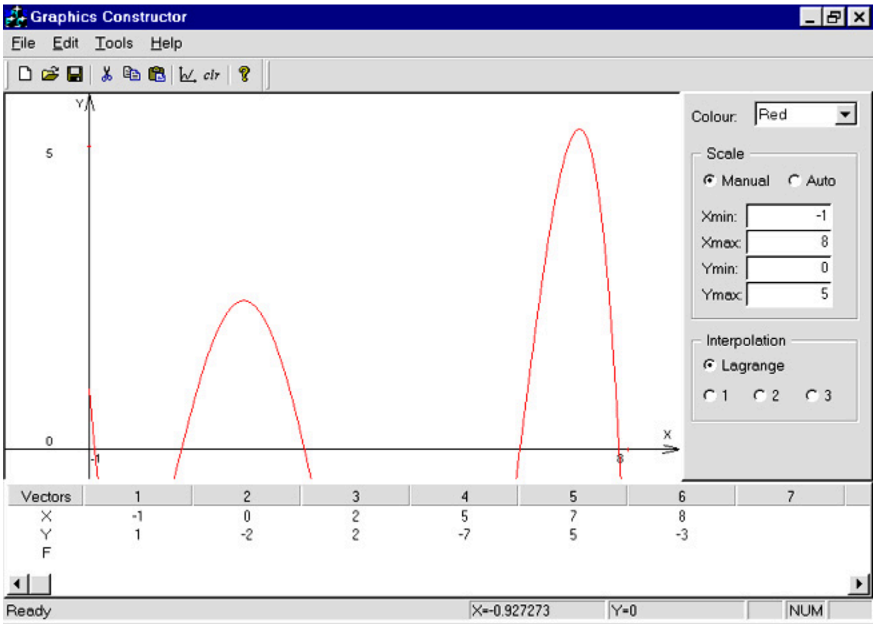

Fig. 5.

and $(Xmax, Ymin)$ are marked red. Co-ordinates of the point under mouse pointer are shown in the status bar panes.

7. Roots approximations

Problem 7.1 Using Table 9 find approximations to the roots of its Lagrangian interpolating polynomial on the interval $(-1, 8)$.

Vectors	0	1	2	3	4	5
X	-1	0	2	5	7	8
Y	1	-2	2	-7	5	-3

Table 9

Solution. Using Table 9 fill in Table "Vectors" of the GC2. Choosing "Lagrange" under "Interpolation", "Blue" in the "Colour" list and "Manual" in the "Scale" with values of $Xmin = -1$, $Xmax = 8$, $Ymin = 0$, $Ymax = 1$, initiate computing. The result is given on Fig. 5.

Using the mouse we can read in the status bar

$$x_1 = -0.910359; x_2 = 0.541833; x_3 = 2.60359;$$

$$x_4 = 6.18924; x_5 = 7.85657.$$

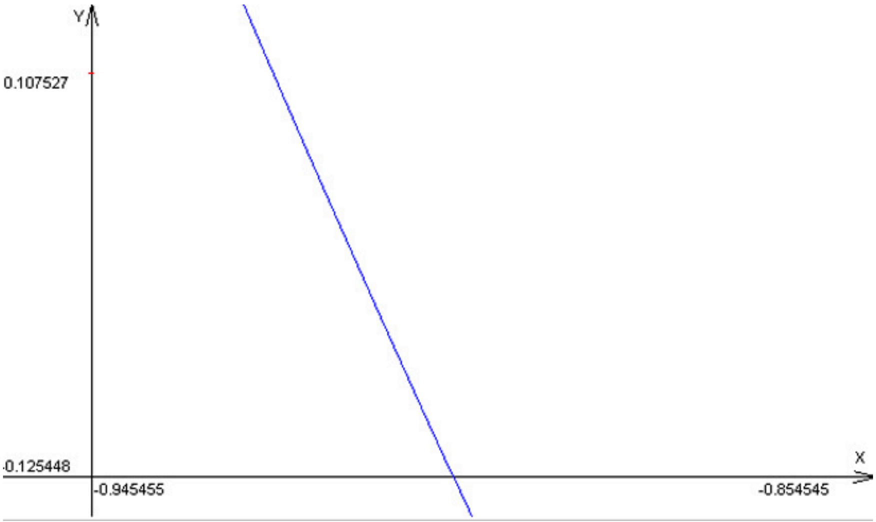

Fig. 6.

- five approximations in the question.

Using "Set Area" in menu "Tools" and the mouse we can refine, for example, approximation of the first root (see Fig. 6).

8. GC2 and definite integrals

Problem 8.1. Using GC2 we can find approximations to the integral

$$\int_0^4 (3x^2 + 2x + 4)dx = 96.$$

Solution. Take Table 10.

X	0	1	2	3	4
Y	4	9	20	37	60

Table 10

The first degree spline (see its graphics on Fig. 7) gives us 98 as a value on "Integral" box.

Since the first derivative of function $3x^2+2x+4$ at the point $x = 0$ is equal to 2 , we can work with Table 11.

X	0	1	2	3	4
Y	4	9	20	37	60
F	2				

Table 11

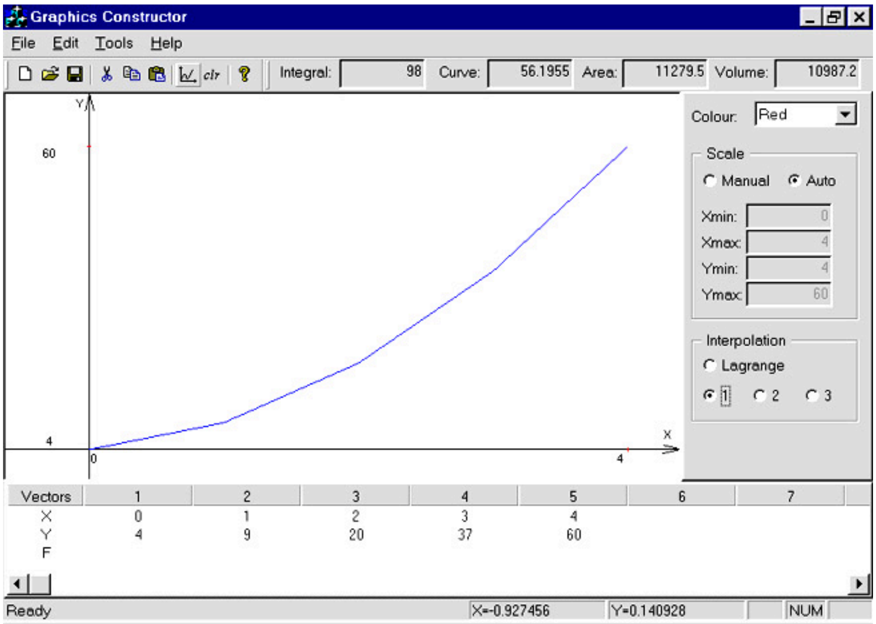
Graphics Constructor
File Edit Tools Help
Integral: 98 Curve: 56.1955 Area: 11279.5 Volume: 10987.2
Colour: Red
Scale
Manual Auto
Xmin: 0
Xmax: 4
Ymin: 4
Ymax: 60
Interpolation
Lagrange
1 2 3
Vectors 1 2 3 4 5 6 7
X 0 1 2 3 4
Y 4 9 20 37 60
F
Ready X=-0.927456 Y=0.140928 NUM

Fig. 7.

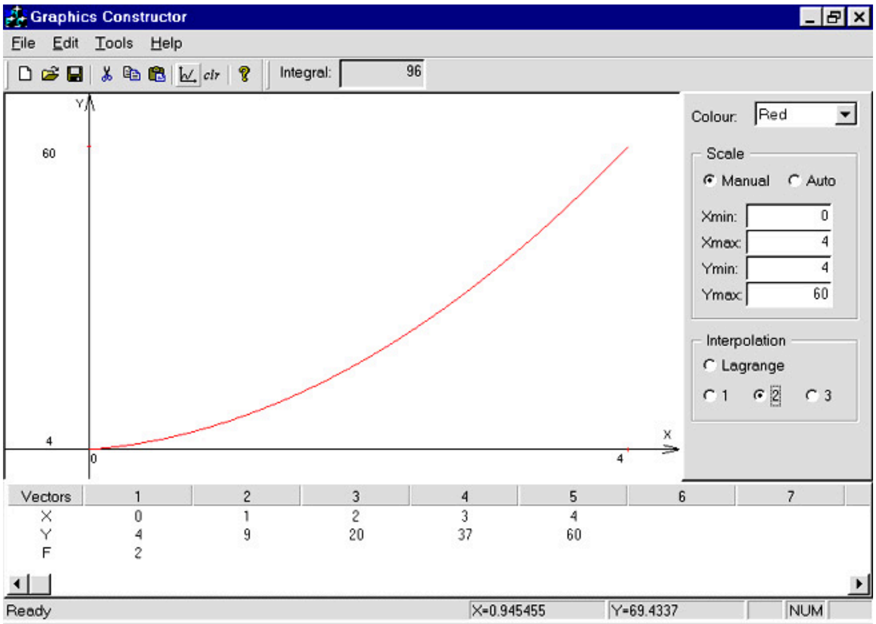
Graphics Constructor
File Edit Tools Help
Integral: 96
Colour: Red
Scale
Manual Auto
Xmin: 0
Xmax: 4
Ymin: 4
Ymax: 60
Interpolation
Lagrange
1 2 3
Vectors 1 2 3 4 5 6 7
X 0 1 2 3 4
Y 4 9 20 37 60
F 2
Ready X=0.945455 Y=69.4337 NUM

Fig. 8.

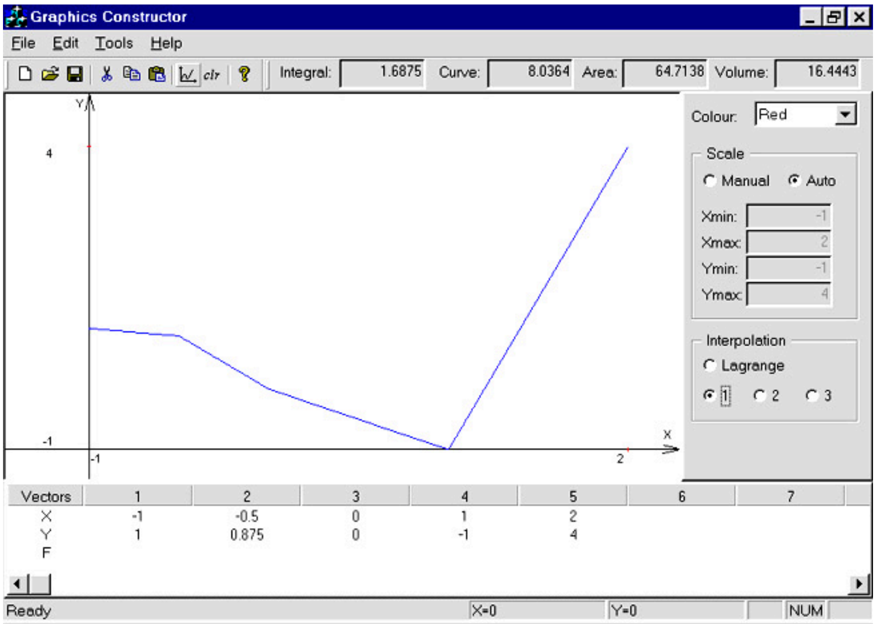

Fig. 9.

The quadratic spline (see its graphics on Fig. 8) gives us 96 as a value on "Integral" box.

Problem 8.2. Using GC2 find approximations

- to the definite integral $\int_{-1}^{2}(x^3 - 2x)dx = 0.75$,

- to the volume and area of the solid obtained by rotating $y = x^3 - 2x$ between $x = -1$, $x = 2$ around the x-axis,

- to the length of the curve $y = x^3 - 2x$, $-1 \leq x \leq 2$.

Solution. Table "Vectors" is filled as Table 12.

X	-1	-0.5	0	1	2
Y	1	0.875	0	-1	4

Table 12

Choosing "1" in "Interpolation", initiate execution. GC2 gives the result given on Fig. 9.

Choosing "Lagrange" in "Interpolation" and "Manual" in "Scale", initiate execution. GC2 gives the result given on Fig. 10.

Now we can refine the computed values. Adding, for example, two points $(0.5, -0.875)$ and $(1.5, 0.375)$ to Table 12, we have Table 13.

X	-1	-0.5	0	0.5	1	1.5	2
Y	1	0.875	0	-0.875	-1	0.375	4

Table 13

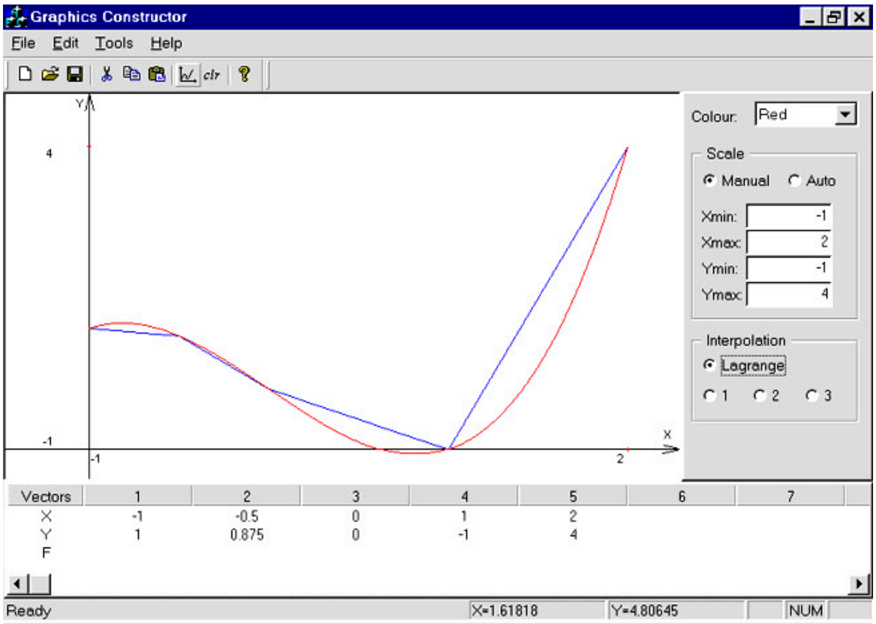

Fig. 10.

Choosing "1" in "Interpolation" and "Auto" in "Scale", initiate execution. GC2 gives the result given on Fig. 11.

9. Conclusion

The textbook "Analysis"[1], which give an evolution of some basic concepts of mathematical analysis, is prepared in the Selcuk University Research Center of Applied Mathematics. It is supported by computer dialogue program Graphics Constructor 2.0.

Acknowledgment. We would like to thank Haydar Bulgak (Selçuk University) for a number of productive discussions about computer - aided methods of studying mathematical analysis.

References

1. Aydın, K., Bulgak, A., and Bulgak, H. (2000): *Analysis* [in Turkish], SelUn Basim Evi, Konya.
2. Babenko, K. I. (1986): *Base of Numerical Analysis* [in Russian], Nauka, Moscow.
3. Bulgak, H. and Eminov D. (2001): Computer dialogue system MVC, Selçuk J. Appl. Math., **2**, No. 2, 17–38.

4. Eminov, D. (2000): Graphics Constructor [in Turkish], in: Analysis, K. Aydın, A. Bulgak, and H. Bulgak (Auths.), SelUn Basim Evi, Konya, 342–358.
5. Estep, D. (2002): *Practical analysis in one variable.* Undergraduate Texts in Mathematics, XX. New-York, Springer.
6. Golub, G. and Ortega, J. (1992): *Scientific computing and differential equations: an introduction to numerical methods.* New-York, Academic Press.
7. Heck, A. (2003): *Introduction to Maple.* 3rd ed., New-York, Springer.
8. Kattan, P. (2003): *MATLAB guide to finite elements: an interactive approach.* X, With CD-ROM. New-York, Springer.
9. Kvasov, B. I. (1998) : *Isogeometrical approximation by splines* [in Russian], Novosibirsk University, Novosibirsk.
10. Kulisch, U. and Miranker W. L. (1981): *Computer numeric in theory and practice.* New-York, Academic Press.
11. Lang, S. (1998): *A first course in calculus.* Undergraduate Texts in Mathematics, New-York, Springer.
12. Philips, G. M. (2003): *Interpolation and approximation by polynomials.* GMS Books in Mathematics, Vol. 14, New-York, Springer.
13. Quarteroni, A. and Saleri, F. (2003): *Scientific computing with MATLAB.*Texts in Computational Science and Engineering, Vol. 2, New-York, Springer.
14. Rozhenko, A. I. (1999): *Abstract theory of splines* [in Russian], Novosibirsk University, Novosibirsk.
15. Trefethen, L. N. (2000): *Spectral methods in MATLAB*, SIAM.
16. Volkov, E. A. (1982): *Numerical methods* [in Russian], Nauka, Moscow.

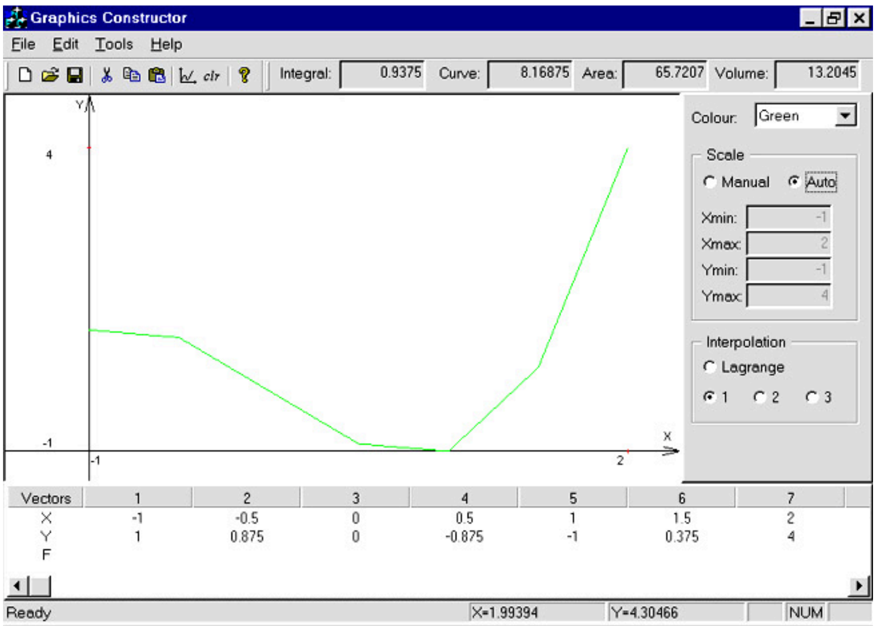

Fig. 11.