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Inverse problems for the kinetic equation of plasma physics and a uniqueness theorem*

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Summary. In this paper we consider the kinetic equation of plasma physics. A uniqueness theorem for an inverse problem for this equation is proved.

Key words: kinetic equation, inverse problem

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In this paper we consider the kinetic equation

$$(1)\qquad \frac{\partial w}{\partial t} + p^1\frac{\partial w}{\partial x^1} + p^2\frac{\partial w}{\partial x^2} + p^3\frac{\partial w}{\partial x^3}$$
$$+ \left(E^1 + p^2B^3 - p^3B^2\right)\frac{\partial w}{\partial p^1} + \left(E^2 + p^3B^1 - p^1B^3\right)\frac{\partial w}{\partial p^2}$$
$$+ \left(E^3 + p^1B^2 - p^2B^1\right)\frac{\partial w}{\partial p^3} = \lambda,$$

where

$$w = w(t,\overline{x},\overline{p}),\quad E^i = E^i(t,\overline{x}),\quad B^i = B^i(t,\overline{x}),\quad i = 1,2,3,$$
$$\overline{x} = (x^1,x^2,x^3),\quad \overline{p} = (p^1,p^2,p^3),\quad \lambda = \lambda(t,\overline{x},\overline{p}).$$

The equation (1) connects (see [1], p. 146) the distribution function w of particles in the space $R^4(t,\overline{x})$ with the electromagnetic field $\overline{E} = \left(E^1,E^2,E^3\right)$, $\overline{B} = \left(B^1,B^2,B^3\right)$.

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The variables t, $\overline{x}$, $\overline{p}$ belong to the domain

$$Q = \{(t, \overline{x}, \overline{p}) \mid \quad |t - t_0| < b,$$

$$|x^i - x_0^i| < a^i, \quad |p^i - p_0^i| < b^i, \quad i = 1, 2, 3\},$$

where x_0^i, p_0^i, a^i, b^i, b, t_0 are constants, $i = 1, 2, 3$.

Consider the following *inverse problem* (see [2]): To find functions $w = w(t, \overline{x}, \overline{p})$, $\lambda = \lambda(t, \overline{x}, \overline{p})$ in the domain Q if $(\overline{E}, \overline{B}) \in C^2(Q)$ and the trace of the function w on the boundary Γ of the domain Q is known, i. e. $w|_\Gamma = w_0(t, \overline{x}, \overline{p})$, $(t, \overline{x}, \overline{p}) \in \Gamma$, where w_0 is a given function.

The following theorem is proved.

Theorem 1. *If $\lambda = \lambda(t, \overline{x}, \overline{p})$ is a solution to the equation*

$$\frac{\partial^2 \lambda}{\partial x^1 \partial p^1} + \frac{\partial^2 \lambda}{\partial x^2 \partial p^2} + \frac{\partial^2 \lambda}{\partial x^3 \partial p^3} = 0 \tag{2}$$

in the domain Q and the quadratic form

$$\sum_{i=1}^{3} (z^i)^2 + \left(\overline{B}, \overline{y}, \overline{z}\right) + \frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial E^j}{\partial x^i} + \frac{\partial E^i}{\partial x^j} \right) y^i y^j \tag{3}$$

$$+ \sum_{i=1}^{3} (y^i)^2 \frac{\partial}{\partial x^i} \left(\overline{p} \times \overline{B}\right)^i + \sum_{i=1}^{3} \frac{\partial B^i}{\partial x^i} y^i (\overline{y} \times \overline{p})^i$$

is positive definite then the inverse problem has no more one solution $w = w(t, \overline{x}, \overline{p})$, $\lambda = \lambda(t, \overline{x}, \overline{p}) \in C^2(Q)$. Here $\overline{y} = (y^1, y^2, y^3)$, $\overline{z} = (z^1, z^2, z^3)$ are independent variables, $\times$ is the symbol of vector product, $(\overline{a})^i$ is the ith component of a vector $(\overline{a})$, $\left(\overline{B}, \overline{y}, \overline{z}\right)$ is mixed product.

In particular, conclusion of the theorem is valid in the following cases:

1. *The matrix*

$$J = \frac{1}{2} \left(\frac{\partial E^i}{\partial x^j} + \frac{\partial E^j}{\partial x^i} \right)_{i,j=1}^{3}$$

is positive definite and

$$\left| \sum_{i=1}^{3} (y^i)^2 \frac{\partial}{\partial x^i} \left(\overline{p} \times \overline{B}\right)^i + \sum_{i=1}^{3} \frac{\partial B^i}{\partial x^i} y^i (\overline{y} \times \overline{p})^i + \left(\overline{B}, \overline{y}, \overline{z}\right) \right| \tag{4}$$

$$< \mu_0|\overline{y}|^2 + |\overline{z}|^2,$$

where μ_0 *is the minimal eigenvalue of the matrix* J *in the domain* Q. *(If* $\overline{B}$ *is a constant vector, then the inequality (4) has the following form* $\left|\left(\overline{B}, \overline{y}, \overline{z}\right)\right| < \mu_0|\overline{y}|^2 + |\overline{z}|^2$.*)*

2. *The vector* $\overline{B} = 0$ *and the matrix* J *is positive definite. (Remark: If the field* $\overline{E}$ *is potential (i. e.* $\overline{E} = \nabla_{\overline{x}}\varphi$ *and* $d^2\varphi > 0$ *in the domain* Q *for a function* $\varphi = \varphi(t, \overline{x}) \in C^2(Q)$*) then the matrix* J *is positive definite. Here* $\nabla_{\overline{x}}\varphi = \left(\frac{\partial\varphi}{\partial x^1}, \frac{\partial\varphi}{\partial x^2}, \frac{\partial\varphi}{\partial x^3}\right)$ *is gradient).*

Proof. Let (w, λ) be a solution to the inverse problem and $w|_\Gamma = 0$. We show that $w = \lambda = 0$ in the domain Q.

Denote the left-hand side of the equation (1) by Λ. Consider the following equality

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial \Lambda}{\partial p^i} = \sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial \lambda}{\partial p^i}, \tag{5}$$

and transform its left-hand and right-hand sides.

The right-hand side of the equality (5) has the form

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial \lambda}{\partial p^i} = \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\left(w \frac{\partial \lambda}{\partial p^i}\right) - w \sum_{i=1}^{3} \frac{\partial^2 \lambda}{\partial x^i \partial p^i}.$$

Taking into account the relation (2) we obtain the following equation

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial \lambda}{\partial p^i} = \sum_{i=1}^{3} \frac{\partial}{\partial x^i}\left(w \frac{\partial \lambda}{\partial p^i}\right). \tag{6}$$

Note that the right-hand side of this equality has divergence form and the functions

$$w \frac{\partial \lambda}{\partial p^i}$$

vanish on the boundary Γ of the domain Q.

Consider the left-hand side of the equation (5). To transform the sum

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial^2 w}{\partial t \partial x^i}$$

we use the following identity

$$\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial y \partial z} \equiv \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)\right], \tag{7}$$

which holds for an arbitrary function $u = u(x, y, z) \in C^2$. Thus,

$$(8)\qquad \sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial^2 w}{\partial t \partial p^i} = \frac{1}{2} \sum_{i=1}^{3} \left[\frac{\partial}{\partial t} \left(\frac{\partial w}{\partial x^i} \frac{\partial w}{\partial p^i} \right) \right.$$
$$\left. - \frac{\partial}{\partial x^i} \left(\frac{\partial w}{\partial t} \frac{\partial w}{\partial p^i} \right) + \frac{\partial}{\partial p^i} \left(\frac{\partial w}{\partial t} \frac{\partial w}{\partial x^i} \right) \right].$$

The right-hand side of the equation (8) has divergence form, and the functions

$$\frac{\partial w}{\partial x^i} \frac{\partial w}{\partial p^i}, \quad \frac{\partial w}{\partial t} \frac{\partial w}{\partial p^i}, \quad \frac{\partial w}{\partial t} \frac{\partial w}{\partial x^i}$$

vanish for $t = t_0 \pm b$, $x^i = x_0^i \pm a^i$, $p^i = p_0^i \pm b^i$ respectively.

Further,

$$(9)\qquad \sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial}{\partial p^i} \left(\sum_{j=1}^{3} p^j \frac{\partial w}{\partial p^j} \right) = \frac{1}{2} \sum_{i=1}^{3} \left(\frac{\partial w}{\partial x^i} \right)^2$$
$$+ \frac{1}{2} \sum_{i=1}^{3} \frac{\partial}{\partial p^i} \left(p^i \left(\frac{\partial w}{\partial x^i} \right)^2 \right) + \sum_{i=1}^{3} p^i \sum_{j=1, j \neq i}^{3} \frac{\partial w}{\partial x^j} \frac{\partial^2 w}{\partial x^i \partial p^j}.$$

The sum

$$\sum_{i=1}^{3} \frac{\partial}{\partial p^i} \left(p^i \left(\frac{\partial w}{\partial x^i} \right)^2 \right)$$

has divergence form, and the functions

$$p^i \left(\frac{\partial w}{\partial x^i} \right)^2$$

vanish for $p^i = p_0^i \pm b^i$.

Taking into account the identity (7) and the condition $j \neq i$, we transform the sum

$$\sum_{i=1}^{3} p^i \sum_{j=1, j \neq i}^{3} \frac{\partial w}{\partial x^j} \frac{\partial^2 w}{\partial x^i \partial p^j}$$

into divergence form, where the functions

$$\frac{\partial w}{\partial x^j} p^i \frac{\partial w}{\partial x^i}, \quad \frac{\partial w}{\partial x^j} p^i \frac{\partial w}{\partial p^j}, \quad \frac{\partial w}{\partial x^i} p^i \frac{\partial w}{\partial p^j}$$

vanish for $p^j = p_0^j \pm b^i$, $x^i = x_0^i \pm a^i$, $x^j = x_0^j \pm a^j$.

To transform the sum

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial}{\partial p^i} \left(\sum_{j=1}^{3} E^j \frac{\partial w}{\partial p^j} \right)$$

we use the identities

$$\frac{\partial u}{\partial x} \frac{\partial}{\partial p} \left(v \frac{\partial u}{\partial p} \right) \equiv \frac{\partial}{\partial p} \left(v \frac{\partial u}{\partial x} \frac{\partial u}{\partial p} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(v \left(\frac{\partial u}{\partial p} \right)^2 \right) + \frac{1}{2} \frac{\partial v}{\partial x} \left(\frac{\partial u}{\partial p} \right)^2, \tag{10}$$

$$\frac{\partial u}{\partial x} \frac{\partial}{\partial p} \left(v \frac{\partial u}{\partial y} \right) \equiv \frac{1}{2} \frac{\partial v}{\partial x} \frac{\partial u}{\partial y} \frac{\partial u}{\partial p} + \frac{1}{2} \left\{ \frac{\partial}{\partial y} \left(v \frac{\partial u}{\partial x} \frac{\partial u}{\partial p} \right) - \frac{\partial}{\partial x} \left(v \frac{\partial u}{\partial y} \frac{\partial u}{\partial p} \right) + \frac{\partial}{\partial p} \left(v \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right) \right\}, \tag{11}$$

which hold for arbitrary functions $u = u(x, y, p)$, $v = v(x) \in C^2$. We obtain the following equation

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial}{\partial p^i} \left(\sum_{j=1}^{3} E^j \frac{\partial w}{\partial p^j} \right) = \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial E^j}{\partial x^i} \frac{\partial w}{\partial p^i} \frac{\partial w}{\partial p^j} + \{\text{divergence terms}\}, \tag{12}$$

where the divergence terms have the form

$$\frac{\partial}{\partial p^i} \left(E^j \frac{\partial w}{\partial x^k} \frac{\partial w}{\partial p^s} \right), \qquad \frac{\partial}{\partial x^i} \left(E^j \frac{\partial w}{\partial p^k} \frac{\partial w}{\partial p^s} \right).$$

To transform the sum

$$\sum_{i=1}^{3} \frac{\partial w}{\partial x^i} \frac{\partial}{\partial p^i} \left(\sum_{j=1}^{3} (\overline{p} \times \overline{B})^j \frac{\partial w}{\partial p^j} \right),$$

we use the identities

$$\frac{\partial u}{\partial x} \frac{\partial}{\partial p} \left(yv \frac{\partial u}{\partial p} \right) \equiv \frac{y}{2} \frac{\partial v}{\partial x} \left(\frac{\partial u}{\partial p} \right)^2 + \frac{\partial}{\partial p} \left(yv \frac{\partial u}{\partial x} \frac{\partial u}{\partial p} \right) - \frac{1}{2} \frac{\partial}{\partial x} \left(yv \left(\frac{\partial u}{\partial p} \right)^2 \right), \tag{13}$$

$$\frac{\partial u}{\partial x}\frac{\partial}{\partial p}\left(pv\frac{\partial u}{\partial y}\right) \equiv \frac{v}{2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} \tag{14}$$

$$+\frac{1}{2}\left\{\frac{\partial}{\partial y}\left(pv\frac{\partial u}{\partial x}\frac{\partial u}{\partial p}\right) - \frac{\partial}{\partial x}\left(pv\frac{\partial u}{\partial y}\frac{\partial u}{\partial p}\right) + \frac{\partial}{\partial p}\left(pv\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\right)\right\},$$

which hold for arbitrary functions $u = u(x, y, p)$, $v = v(x) \in C^2$. We obtain the following equation

$$2\sum_{i=1}^{3}\frac{\partial w}{\partial x^i}\frac{\partial}{\partial p^i}\left(\sum_{j=1}^{3}\left(\overline{p}\times\overline{B}\right)^j\frac{\partial w}{\partial p^j}\right) \tag{15}$$

$$= \left(\frac{\partial w}{\partial p^1}\right)^2\left(p^2\frac{\partial B^3}{\partial x^1} - p^3\frac{\partial B^2}{\partial x^1}\right) + \left(\frac{\partial w}{\partial p^2}\right)^2\left(p^3\frac{\partial B^1}{\partial x^2} - p^1\frac{\partial B^3}{\partial x^2}\right)$$

$$+\left(\frac{\partial w}{\partial p^3}\right)^2\left(p^1\frac{\partial B^2}{\partial x^3} - p^2\frac{\partial B^1}{\partial x^3}\right)$$

$$+\frac{\partial w}{\partial p^2}\frac{\partial w}{\partial p^3}p^1\left(\frac{\partial B^2}{\partial x^2} - \frac{\partial B^3}{\partial x^3}\right) + \frac{\partial w}{\partial p^1}\frac{\partial w}{\partial p^3}p^2\left(\frac{\partial B^3}{\partial x^3} - \frac{\partial B^1}{\partial x^1}\right)$$

$$+\frac{\partial w}{\partial p^1}\frac{\partial w}{\partial p^2}p^3\left(\frac{\partial B^1}{\partial x^1} - \frac{\partial B^2}{\partial x^2}\right) + \{\text{divergence terms}\},$$

where the divergence terms have the form

$$\frac{\partial}{\partial p^i}\left(p^jB^k\frac{\partial w}{\partial x^s}\frac{\partial w}{\partial p^m}\right), \qquad \frac{\partial}{\partial x^i}\left(p^jB^k\frac{\partial w}{\partial p^s}\frac{\partial w}{\partial p^m}\right).$$

Thus we have

$$\sum_{i=1}^{3}\frac{\partial w}{\partial x^i}\frac{\partial \Lambda}{\partial p^i} = \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial w}{\partial x^i}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{3}\frac{\partial E^j}{\partial x^i}\frac{\partial w}{\partial p^i}\frac{\partial w}{\partial p^j} \tag{16}$$

$$+\frac{1}{2}\left(\overline{B}, \nabla_{\overline{p}}w, \nabla_{\overline{x}}w\right) + \frac{1}{2}\sum_{i=1}^{3}\left(\frac{\partial w}{\partial p^i}\right)^2\frac{\partial}{\partial x^i}\left(\overline{p}\times\overline{B}\right)^i$$

$$+\frac{1}{2}\frac{\partial B^i}{\partial x^i}\frac{\partial w}{\partial p^i}\left(\nabla_{\overline{p}}w\times\overline{p}\right)^i + \{\text{divergence terms}\}.$$

Integrating (5) on the domain Q, taking into account the equations (6), (16) and the form of the divergence terms, we obtain the following equation

$$\int_Q\left[\sum_{i=1}^{3}\left(\frac{\partial w}{\partial x^i}\right)^2 + \frac{1}{2}\sum_{i,j=1}^{3}\left(\frac{\partial E^j}{\partial x^i} + \frac{\partial E^i}{\partial x^j}\right)\frac{\partial w}{\partial p^i}\frac{\partial w}{\partial p^j}\right. \tag{17}$$

$$+\left(\overline{B},\nabla_{\overline{p}}w,\nabla_{\overline{x}}w\right)+\sum_{i=1}^{3}\left(\frac{\partial w}{\partial p^i}\right)^2\frac{\partial}{\partial x^i}\left(\overline{p}\times\overline{B}\right)^i$$

$$\left.+\sum_{i=1}^{3}\frac{\partial B^i}{\partial x^i}\frac{\partial w}{\partial p^i}(\nabla_{\overline{p}}w\times\overline{p})^i\right]dtd\overline{x}d\overline{p}=0.$$

According to the conditions of the theorem (see (3)), the quadratic form

$$\sum_{i=1}^{3}\left(\frac{\partial w}{\partial x^i}\right)^2+\left(\overline{B},\nabla_{\overline{p}}w,\nabla_{\overline{x}}w\right)$$

$$+\frac{1}{2}\sum_{i,j=1}^{3}\left(\frac{\partial E^j}{\partial x^i}+\frac{\partial E^i}{\partial x^j}\right)\frac{\partial w}{\partial p^i}\frac{\partial w}{\partial p^j}$$

$$+\sum_{i=1}^{3}\left(\frac{\partial w}{\partial p^i}\right)^2\frac{\partial}{\partial x^i}\left(\overline{p}\times\overline{B}\right)^i+\sum_{i=1}^{3}\frac{\partial B^i}{\partial x^i}\frac{\partial w}{\partial p^i}(\nabla_{\overline{p}}w\times\overline{p})^i$$

is positive definite. Consequently, the equality (17) holds if and only if $\nabla_{\overline{p}}w=0$, $\nabla_{\overline{x}}w=0$. Since $w|_{\Gamma}=0$ it follows that $w|_Q=0$. From the equation (1) we obtain that $\lambda=0$.

This completes the proof. □

Remark 1. From the physical point of view the supposition about truth of the inequality (4) is quite justified because $\overline{B}$ includes a factor $1/c$, where c is velocity of light.

Remark 2. The theorem will be also true if we put λw instead of λ in the right-hand side of the equation (1). From the physical point of view these problems describe processes of absorption and radiation in the considered domain.

Remark 3. We can consider the domain Q in a more general form

$$Q=\left\{(t,\overline{x},\overline{p})\mid\quad |t-t_0|<b,\ \overline{x}\in D_1,\ \overline{p}\in D_2\right\},$$

where D_1, D_2 are domains in the spaces $R^3(\overline{x})$, $R^3(\overline{p})$ respectively. In this case the theorem will be true too.

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